III. TWO TANGENT SPACE FORMALISM: NOT ALL INDICES ARE CREATED EQUAL

1. Introduction

In Sections I.2 and II.5, I argued for the introduction into general relativity of an independent full connection, $\Gamma^{\alpha}{}_{\beta\gamma}$, in addition to the Christoffel connection, $\{^{\alpha}{}_{\beta\gamma}\}$. The Christoffel connection is metric-compatible and torsion-free, while depending on the theory, the full connection may be non-metric-compatible and/or non-torsion-free. The difference between the connections,

$$\lambda^{\alpha}{}_{\beta\gamma} = \Gamma^{\alpha}{}_{\beta\gamma} - \{^{\alpha}{}_{\beta\gamma}\},$$

is the defect tensor.

Again depending on the theory, the gauge group, G, of the full connection can be any one of the groups listed in Tables II.2 and II.4 (or another group not listed). Thus if the frame field is chosen from the corresponding class of frames (listed in Tables II.1 and II.3) then the connection coefficients take their values in the Lie algebra of the gauge group as listed in Table II.5.

In such theories with an independent connection, it is often convenient to express the Bianchi identities, conservation laws and field equations using "mixed" covariant derivatives such as

$$\nabla_{\gamma}\, t_{\hat{\alpha}}{}^{\tilde{\beta}} = e_{\gamma}\, t_{\hat{\alpha}}{}^{\tilde{\beta}} - \Gamma^{\hat{\mu}}{}_{\hat{\alpha}\gamma}\, t_{\hat{\mu}}{}^{\tilde{\beta}} + \{^{\tilde{\beta}}{}_{\tilde{\mu}\gamma}\}\, t_{\hat{\alpha}}{}^{\tilde{\mu}} .$$

These are "mixed" because some indices (those with a caret, ^,) are corrected with the full connection, while other indices (those with a tilde, ~,) are corrected with the Christoffel connection. Of course, it is always possible to use only full connections or only Christoffel connections, but then the equations contain unaesthetic defect tensors.

This chapter presents a formalism which makes rigorous the concepts of two types of indices and mixed covariant derivatives. This is accomplished by introducing two isomorphic copies of the tangent bundle (distinguished as internal and external) each with its own connection.

The external tangent bundle, TM, is the usual tangent bundle with the Christoffel connection. Its bundle structure is defined using coordinate bases but arbitrary frame fields are admissible.

The internal tangent bundle, $\hat{T}M$, is an abstract 4-dimensional vector bundle associated, by some representation, R_T, to some principle G-bundle, P, with a G-connection. (Here, G could be one of the groups listed in Tables II.2 and II.4.) Its bundle structure is defined using frame fields which are compatible with the group G in that the bundle of G-compatible frames is a principal $R_T(G)$-bundle associated to P. (For each G, the class of G-compatible frames and the G-compatible frame bundle are listed in Tables II.1 and II.3.)

In Section 2, I discuss the two tangent spaces, their frame fields and connections and the isomorphism between them. The covariant derivative of this isomorphism turns out to be the defect tensor. If there is a metric on both tangent spaces, then the isomorphism is required to be an isometry. (Note that a bimetric theory of gravity could easily be incorporated into this formalism by not requiring the isomorphism to be an isometry.)

In section 3, I compare the two tangent space formalism with the standard one tangent space formalism in both the index notation and the Cartan differential form notation. In section 4, I discuss minimal coupling of the matter Lagrangian and the related question of which indices are internal or external. This choice is the same as that made in the Cartan form notation between visible and invisible indices. The invisible form indices become external while the visible tensor indices become internal. Thus most matter fields are assumed to be cross sections of internal tangent tensor bundles. However, gauge fields are Lie algebra valued external 1-forms and differentiations are standardly performed in external vector directions.

In section 5, I give an example of a computation using the two tangent space formalism by deriving energy-momentum and angular momentum conservation laws via Noether's theorem. I derive the conservation laws when the internal tangent gauge group is $GL_\circ(4,R)$ and $O_\circ(3,1,R)$, but it is obvious that a similar derivation could be done for any tangent gauge group. In the $GL_\circ(4,R)$ case I also obtain conservation laws for hypermomentum and dilation current. Section 5 also serves to establish terminology for later chapters.

Throughout this chapter, it is useful to keep in mind the relationship between the spinor bundle, $T^{\circ\circ}_{\frac{1}{2}\frac{1}{2}}M_{Herm}$, and the vector bundle, $T^{\circ}_{1}M$, as discussed in Section II.3 I regard that relationship to be a special case of the relationship between $\hat{T}M$ and TM discussed in this chapter.

2. Two Tangent Space Formalism

Let TM, T^*M, and T^q_pM be the tangent bundle, cotangent bundle and various tensor bundles over a spacetime M. Hereafter, TM, T^*M and T^q_pM will be referred to as the <u>external tangent bundle</u>, <u>external cotangent bundle</u> and <u>external tangent tensor bundles</u>. Their elements are <u>external tangent vectors</u>, <u>external tangent covectors</u> and <u>external tangent tensors</u>, which carry <u>external tangent indices</u>.

Let G be a Lie group which has a 4-dimensional real representation R_T. (Some interesting groups G are listed in Tables II.2 and II.4.) Let P be a principal fibre bundle over M with gauge group G. (Tables II.1 and II.3 list a frame bundle P for each group G.) Let $\hat{T}M$ be the R^4 vector bundle over M associated to P by the representation R_T. Let $\hat{T}^*M$ be its dual bundle and let $\hat{T}^s_rM$ be the various tensor bundles formed from $\hat{T}M$ and $\hat{T}^*M$. Hereafter, $\hat{T}M$, $\hat{T}^*M$ and $\hat{T}^s_rM$ will be referred to as the <u>internal tangent bundle</u>, <u>internal cotangent bundle</u> and <u>internal tangent tensor bundles</u>. Their elements are <u>internal tangent vectors</u>, <u>internal tangent covectors</u> and <u>internal tangent tensors</u>, which carry <u>internal tangent indices</u>.

To distinguish between internal and external indices, I will put a caret (^) over the internal indices and tilde (~) over external indices. One can also construct the <u>mixed tensor bundles</u> $T^{\tilde{q}\hat{s}}_{\tilde{p}\hat{r}}M = T^q_pM \otimes \hat{T}^s_rM$ whose elements are <u>mixed tensors</u>.

Up to this point, there is no justification for calling $\hat{T}M$ a tangent bundle. I now assume there is a vector bundle isomorphism, $\sigma : \hat{T}M \rightarrow TM$, hereafter called the <u>soldering isomorphism</u>. Thus, $\hat{T}M$ is identical to TM as a vector bundle and hence deserves the name tangent bundle. However, I will distinguish between $\hat{T}M$ and TM by two additional structures: the class of admissible frames and the choice of connection.

The first distinguishing structure is the class of admissible frames. For TM, any frame, $e_{\tilde{\alpha}}$, is admissible. However, for $\hat{T}M$, the only admissible frames, $u_{\hat{\alpha}}$, are those which are "compatible" with the group G. The definition of a G-compatible frame must be made separately for each G. It is chosen so that at each point of M, the set of G-compatible frames is homeomorphic to $R_T(G)$, the image under the representation, R_T, of the group G. This is a restriction iff $R_T(G) \neq GL(4,R)$. (Tables II.1 and II.3 list the class of compatible frames for each group G.) The bundle of G-compatible frames is then a principal $R_T(G)$-bundle which is the image under R_T of the original principal G-bundle, P, to which TM was associated.

Let $\theta^{\tilde{\alpha}}$ be the basis for T^*M dual to $e_{\tilde{\alpha}}$ and let $v^{\hat{\alpha}}$ be the basis for $\hat{T}^*M$ dual to $u_{\hat{\alpha}}$. Various tensor products of $e_{\tilde{\alpha}}$, $\theta^{\tilde{\alpha}}$, $u_{\hat{\alpha}}$ and $v^{\hat{\alpha}}$ provide bases for the various mixed tensor bundles $T^{\tilde{q}\hat{s}}_{\tilde{p}\hat{r}}M$; e.g. if $\psi \in T^{\tilde{1}\hat{1}}_{\tilde{1}\hat{1}}M$, then $\psi = \psi^{\tilde{\alpha}}{}_{\tilde{\beta}}{}^{\hat{\gamma}}{}_{\hat{\delta}}\, e_{\tilde{\alpha}} \otimes \theta^{\tilde{\beta}} \otimes u_{\hat{\gamma}} \otimes v^{\hat{\delta}}$.

The second structure which distinguishes between $\hat{T}M$ and TM is the choice of connection. I assume there is a metric on TM so that the Christoffel connection (metric-compatible, torsion-free) defines a covariant derivative, ∇, on TM, T^*M and T^q_pM. The Christoffel symbols, $\{^{\tilde{\alpha}}_{\tilde{\beta}\tilde{\gamma}}\}$, are defined by

$$\nabla_{e_{\tilde{\gamma}}} e_{\tilde{\beta}} = \{^{\tilde{\alpha}}_{\tilde{\beta}\tilde{\gamma}}\} e_{\tilde{\alpha}}. \tag{1}$$

In general, the Christoffel connection 1-form,

$$\tilde{\omega}^{\tilde{\alpha}}{}_{\tilde{\beta}} = \{^{\tilde{\alpha}}_{\tilde{\beta}\tilde{\gamma}}\}\theta^{\tilde{\gamma}} , \tag{2}$$

takes its values in $\mathcal{L}GL(4,R)$, the Lie algebra of GL(4,R). However, if e_α is an orthonormal frame, then $\tilde{\omega}^{\tilde{\alpha}}{}_{\tilde{\beta}}$ takes its values in $\mathcal{L}O(3,1,R)$, the Lie algebra of O(3,1,R).

I assume there is a connection on the abstract principal G-bundle, P. The connection on P induces a covariant derivative, ∇, on its associated G-vector bundles, including $\hat{T}M$, $\hat{T}^{*}M$ and $\hat{T}^{q}_{p}M$. Its connection coefficients, $\Gamma^{\hat{\alpha}}{}_{\hat{\beta}\tilde{\gamma}}$, are defined by

$$\nabla_{e_{\tilde{\gamma}}} u_{\hat{\beta}} = \Gamma^{\hat{\alpha}}{}_{\hat{\beta}\tilde{\gamma}}\, u_{\hat{\alpha}} \; . \tag{3}$$

The connection 1-form,

$$\hat{\omega}^{\hat{\alpha}}{}_{\hat{\beta}} = \Gamma^{\hat{\alpha}}{}_{\hat{\beta}\tilde{\gamma}}\, \theta^{\tilde{\gamma}} \quad , \tag{4}$$

takes its values in $\mathcal{L}G$, the Lie algebra of the gauge group G. (Table II.5 lists $\mathcal{L}G$ for each G in Tables II.2 and II.4).

The covariant derivatives, ∇ on $T^{q}_{p}M$ and ∇ on $\hat{T}^{s}_{r}M$, extend to a covariant derivative, ∇, on $T^{\tilde{q}\hat{s}}_{\tilde{p}\hat{r}}M$ by demanding that ∇ be Liebnizian. Thus for example, if $\psi \in T^{\tilde{1}\hat{1}}_{\tilde{1}\hat{1}}M$, then

$$\nabla_{\tilde{\varepsilon}} \psi^{\tilde{\alpha}}{}_{\tilde{\beta}}{}^{\hat{\gamma}}{}_{\hat{\delta}} = e_{\tilde{\varepsilon}}(\psi^{\tilde{\alpha}}{}_{\tilde{\beta}}{}^{\hat{\gamma}}{}_{\hat{\delta}}) + \{^{\tilde{\alpha}}{}_{\tilde{\mu}\tilde{\varepsilon}}\}\, \psi^{\tilde{\mu}}{}_{\tilde{\beta}}{}^{\hat{\gamma}}{}_{\hat{\delta}} - \{^{\tilde{\mu}}{}_{\tilde{\beta}\tilde{\varepsilon}}\}\, \psi^{\tilde{\alpha}}{}_{\tilde{\mu}}{}^{\hat{\gamma}}{}_{\hat{\delta}}$$

$$+ \Gamma^{\hat{\gamma}}{}_{\hat{\mu}\tilde{\varepsilon}}\, \psi^{\tilde{\alpha}}{}_{\tilde{\beta}}{}^{\hat{\mu}}{}_{\hat{\delta}} - \Gamma^{\hat{\mu}}{}_{\hat{\delta}\tilde{\varepsilon}}\, \psi^{\tilde{\alpha}}{}_{\tilde{\beta}}{}^{\hat{\gamma}}{}_{\hat{\mu}} \; . \tag{5}$$

Notice that the differentiating direction (X in $\nabla_X \psi$) always belongs to TM. This is true of all covariant derivatives, regardless of the gauge group or whether the vector bundle, in this case TM, $\hat{T}M$ or $T^{\tilde{q}\hat{s}}_{\tilde{p}\hat{r}}M$, is in any sense related to the tangent bundle.

Throughout the rest of this section I discuss the soldering isomorphism, σ, in more detail. First notice that σ is actually the mixed tensor

$$\sigma = \sigma^{\tilde{\alpha}}{}_{\hat{\beta}}\, e_{\tilde{\alpha}} \otimes v^{\hat{\beta}} : \hat{T}M \to TM. \tag{6}$$

The inverse soldering isomorphism is the tensor

$$\sigma^{-1} = (\sigma^{-1})^{\hat{\beta}}{}_{\tilde{\alpha}}\, u_{\hat{\beta}} \otimes \theta^{\tilde{\alpha}} : TM \to \hat{T}M, \tag{7}$$

whose components, $(\sigma^{-1})^{\hat{\beta}}{}_{\tilde{\alpha}}$, are the matrix inverse of $\sigma^{\tilde{\alpha}}{}_{\hat{\beta}}$. The same tensors, σ and σ^{-1}, provide isomorphisms between the dual spaces:

$$\sigma = \sigma^{\tilde{\alpha}}{}_{\hat{\beta}}\, e_{\tilde{\alpha}} \otimes v^{\hat{\beta}} : T^{*}M \to \hat{T}^{*}M, \tag{8}$$

$$\sigma^{-1} = (\sigma^{-1})^{\hat{\beta}}{}_{\tilde{\alpha}}\, u_{\hat{\beta}} \otimes \theta^{\tilde{\alpha}} : \hat{T}^{*}M \to T^{*}M. \tag{9}$$

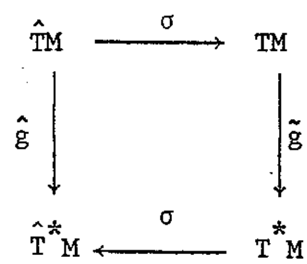
Just as the metric, g, and its inverse, g^{-1}, are used to convert contravariant indices into covariant indices and vice versa, so the soldering isomorphism, σ, and its inverse, σ^{-1}, are used to convert internal indices into external indices and vice versa. Thus the external vector, $X^{\tilde{\alpha}}$, and 1-form, $A_{\tilde{\alpha}}$, are related to the internal vector, $X^{\hat{\beta}}$, and 1-form, $A_{\hat{\beta}}$, by the formulas

$$\begin{aligned} X^{\tilde{\alpha}} &= \sigma^{\tilde{\alpha}}{}_{\hat{\beta}}\, X^{\hat{\beta}}, & A_{\tilde{\alpha}} &= (\sigma^{-1})^{\hat{\beta}}{}_{\tilde{\alpha}}\, A_{\hat{\beta}}, \\ X^{\hat{\beta}} &= (\sigma^{-1})^{\hat{\beta}}{}_{\tilde{\alpha}}\, X^{\tilde{\alpha}}, & A_{\hat{\beta}} &= \sigma^{\tilde{\alpha}}{}_{\hat{\beta}}\, A_{\tilde{\alpha}}. \end{aligned} \tag{10}$$

In particular, the external metric, $g_{\tilde{\alpha}\tilde{\beta}}$, is converted into an internal metric,

$$g_{\hat{\gamma}\hat{\delta}} = \sigma^{\tilde{\alpha}}{}_{\hat{\gamma}}\, \sigma^{\tilde{\beta}}{}_{\hat{\delta}}\, g_{\tilde{\alpha}\tilde{\beta}}. \tag{11}$$

For some of the groups, G, the definition of G-compatible frames requires an internal metric on $\hat{T}M$. I assume that $g_{\hat{\gamma}\hat{\delta}}$ is that metric. In other words, I assume that the isomorphism, σ, is actually an isometry between $\hat{T}M$ with the metric $g_{\hat\gamma\hat\delta}$ and TM with the metric $g_{\tilde\alpha\tilde\beta}$. Equation (11) guarantees that the following diagram commutes:



$$\hat{g} = \sigma \circ \tilde{g} \circ \sigma \, .$$

In other words, raising and lowering of indices commutes with internalizing and externalizing of indices. Equation (11) also gives a formula for the components of σ^{-1}:

$$(\sigma^{-1})^{\hat\beta}{}_{\tilde\alpha} = g_{\tilde\alpha\tilde\gamma}\, \sigma^{\tilde\gamma}{}_{\hat\delta}\, g^{\hat\delta\hat\beta} = \sigma_{\tilde\alpha}{}^{\hat\beta} \, . \tag{12}$$

It should be noticed that this two tangent space formalism could easily be generalized to include bimetric theories of gravity by simply not assuming that σ is an isometry. Then the choice of metric would be a third structure distinguishing between TM and $\hat{T}M$.

Given the G-compatible bases $u_{\hat\alpha}$ and $v^{\hat\alpha}$ on $\hat{T}M$ and $\hat{T}^*M$, the soldering isomorphism, σ, defines associated G-compatible bases on TM and T^*M:

$$e_{\tilde\alpha} = \sigma(u_{\hat\alpha}), \qquad \theta^{\tilde\alpha} = \sigma^{-1}(v^{\hat\alpha}). \tag{13}$$

In associated bases the isomorphisms, σ and σ^{-1}, are the tensors

$$\sigma = e_{\tilde\alpha} \otimes v^{\hat\alpha} \ , \qquad \sigma^{-1} = u_{\hat\alpha} \otimes \theta^{\tilde\alpha} \ , \tag{14}$$

with components

$$\sigma^{\tilde\alpha}{}_{\hat\beta} = \delta^{\tilde\alpha}{}_{\hat\beta} \ , \qquad (\sigma^{-1})^{\hat\beta}{}_{\tilde\alpha} = \delta^{\hat\beta}{}_{\tilde\alpha} \ . \tag{15}$$

Just as orthonormal frames make it easier to raise and lower indices, associated frames make it easier to internalize and externalize indices, especially when listing components of tensors.

Unless $R_T(G) = GL(4,R)$, not every basis on TM is G-compatible. At times one may wish to use a non-G-compatible basis such as a coordinate basis. For future reference, notice that if the isomorphism, σ, and an associated G-compatible basis, $e_{\tilde{\alpha}}$, are each expanded in a coordinate basis, $\partial_{\tilde{a}} = \partial/\partial x^{\tilde{a}}$,

$$\sigma = \sigma^{\tilde{a}}{}_{\hat{\alpha}}\, \partial_{\tilde{a}} \otimes v^{\hat{\alpha}} , \tag{16}$$

$$e_{\tilde{\alpha}} = \sigma(u_{\hat{\alpha}}) = \sigma^{\tilde{a}}{}_{\tilde{\alpha}}\, \partial_{\tilde{a}} , \tag{17}$$

then their components are numerically equal.

Finally, I discuss the covariant derivative of σ:

$$\nabla_{\tilde{\gamma}}\, \sigma^{\tilde{\alpha}}{}_{\hat{\beta}} = e_{\tilde{\gamma}}(\sigma^{\tilde{\alpha}}{}_{\hat{\beta}}) + \{^{\tilde{\alpha}}_{\tilde{\mu}\tilde{\gamma}}\}\sigma^{\tilde{\mu}}{}_{\hat{\beta}} - \Gamma^{\hat{\mu}}{}_{\hat{\beta}\tilde{\gamma}}\, \sigma^{\tilde{\alpha}}{}_{\hat{\mu}} . \tag{18}$$

In associated G-compatible bases,

$$\nabla_{\tilde{\gamma}}\, \sigma^{\tilde{\alpha}}{}_{\hat{\beta}} = \{^{\tilde{\alpha}}_{\hat{\beta}\tilde{\gamma}}\} - \Gamma^{\tilde{\alpha}}{}_{\hat{\beta}\tilde{\gamma}} = - \lambda^{\tilde{\alpha}}{}_{\hat{\beta}\tilde{\gamma}} . \tag{19}$$

Thus the covariant derivative of σ is the negative of the defect tensor. This fact may also be derived invariantly by noticing that there are actually two connections on TM, namely ∇ and $\sigma \circ \nabla \circ \sigma^{-1}$. The difference between these is the defect tensor. Thus,

$$\begin{aligned}\lambda^{\tilde{\alpha}}{}_{\tilde{\beta}\tilde{\gamma}}\, \psi^{\tilde{\beta}} &= \sigma^{\tilde{\alpha}}{}_{\hat{\delta}}\, \nabla_{\tilde{\gamma}}[(\sigma^{-1})^{\hat{\delta}}{}_{\tilde{\beta}}\psi^{\tilde{\beta}}] - \nabla_{\tilde{\gamma}}\psi^{\tilde{\alpha}} \\ &= \sigma^{\tilde{\alpha}}{}_{\hat{\delta}}[\nabla_{\tilde{\gamma}}(\sigma^{-1})^{\hat{\delta}}{}_{\tilde{\beta}}]\psi^{\tilde{\beta}} = - (\nabla_{\tilde{\gamma}}\, \sigma^{\tilde{\alpha}}{}_{\hat{\delta}})\, (\sigma^{-1})^{\hat{\delta}}{}_{\tilde{\beta}}\, \psi^{\tilde{\beta}} .\end{aligned}$$

Therefore

$$\lambda^{\tilde{\alpha}}{}_{\hat{\beta}\tilde{\gamma}} = - \nabla_{\tilde{\gamma}} \sigma^{\tilde{\alpha}}{}_{\hat{\beta}} \; . \tag{20}$$

One says that σ preserves the connection or that the connection is σ-compatible if for all $X \varepsilon TM$ and all $\psi \varepsilon \hat{T}M$,

$$\nabla_X(\sigma\psi) = \sigma \nabla_X \psi \; , \tag{21}$$

or equivalently,

$$0 = \nabla_X \sigma = - \lambda(X). \tag{22}$$

Thus the connection is σ-compatible iff the defect tensor vanishes so that the two connections, ∇ and $\sigma \circ \nabla \circ \sigma^{-1}$, coincide.

3. Comparison with Single Tangent Space Formalism in Index Notation and Cartan Differential Form Notation

Except for the soldering isomorphism, equations in index notation in the two tangent space formalism are identical in appearance with equations in index notation with a single tangent space. The difference is one of philosophy in that the same symbols are interpreted differently. For example in either notation the Bianchi identity for the curvature, $\hat{R}^{\alpha}{}_{\beta\gamma\delta}$, of the full connection, $\Gamma^{\alpha}{}_{\beta\gamma}$, can be written in three ways:

$$0 = \varepsilon^{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}\tilde{\delta}}(\nabla_{\tilde{\beta}}\,\hat{R}^{\tilde{\mu}}{}_{\tilde{\nu}\tilde{\gamma}\tilde{\delta}} + \lambda^{\tilde{\mu}}{}_{\tilde{\kappa}\tilde{\beta}}\,\hat{R}^{\tilde{\kappa}}{}_{\tilde{\nu}\tilde{\gamma}\tilde{\delta}} - \lambda^{\tilde{\kappa}}{}_{\tilde{\nu}\tilde{\beta}}\,\hat{R}^{\tilde{\mu}}{}_{\tilde{\kappa}\tilde{\gamma}\tilde{\delta}}) , \qquad (1)$$

$$0 = \varepsilon^{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}}(\nabla_{\hat{\beta}}\,\hat{R}^{\hat{\mu}}{}_{\hat{\nu}\hat{\gamma}\hat{\delta}} + 2\,\lambda^{\hat{\kappa}}{}_{\hat{\gamma}\hat{\beta}}\,\hat{R}^{\hat{\mu}}{}_{\hat{\nu}\hat{\kappa}\hat{\delta}}) , \qquad (2)$$

$$0 = \varepsilon^{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}\tilde{\delta}}\,\nabla_{\tilde{\beta}}\,\hat{R}^{\hat{\mu}}{}_{\hat{\nu}\tilde{\gamma}\tilde{\delta}} \quad . \qquad (3)$$

Equation (1) has all Christoffel covariant derivatives; (2) has all full covariant derivatives; and (3) has mixed covariant derivatives. In the standard index notation, one would usually establish a convention that all covariant derivatives are either Christoffel or full and then drop the tildas from (1) or the carets from (2) respectively. However one could also use (3) and regard the tildas and carets merely as a notational shorthand for remembering which indices are corrected with the Christoffel connection and which with the full connection.

Notice how much simpler equation (3) appears than equations (1) and (2). The same sort of simplification occurs using mixed covariant derivatives in the Bianchi identity for the torsion, $Q^{\alpha}{}_{\gamma\delta}$; in the Noether conservation laws of energy-momentum and angular-momentum (Section III.5);

and in the field equations of certain theories of gravity with an independent connection (Section V.3d). Why do these equations look simpler using mixed covariant derivatives? Why are some indices corrected with the full connection and some with the Christoffel connection?

Compare equation (3) with the Bianchi identity for a Yang-Mills field in curved spacetime:

$$0 = \varepsilon^{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}\tilde{\delta}} \nabla_{\tilde{\beta}} F^{A}{}_{B\tilde{\gamma}\tilde{\delta}} . \tag{4}$$

Here $F^{A}{}_{B\gamma\delta}$ is the gauge curvature written as a matrix in some representation of the internal symmetry gauge group. Notice that the spacetime indices in (4) all have Christoffel corrections and correspond to the spacetime indices in (3) which have Christoffel corrections. Similarly, the internal indices in (4) correspond to the spacetime indices in (3) which are corrected with full connections.

Furthermore, if one examines the derivations of the conservation laws and of the field equations, one recognizes that the Christoffel corrections always arise through an integration by parts, an operation intimately related to the tangent bundle. The other indices merely go along for the ride. In fact, Stokes' theorem (specifically the divergence theorem) only works for partial derivatives and Christoffel covariant derivatives.

These arguments seem to imply that there are two types of spacetime indices: External spacetime indices are corrected with the Christoffel connection and describe directions in spacetime. Internal spacetime indices are corrected with the full connection, describe components of most fields, and are more analogous to the indices of a gauge theory of an internal symmetry.

However, indices are merely labels for the basis vectors, and with only one tangent space, nothing can distinguish between the internal basis and the external basis. Either they coincide or one can be expanded in terms of the other. Consequently, the philosophy behind the standard index notation has difficulty explaining why equation (3) is simpler than (1) or (2).

On the other hand, the two tangent space formalism gives a geometric explanation: there are two kinds of indices because they label bases living in two different spaces, albeit that the two spaces are isomorphic.

In addition to the change in philosophy, the two tangent space formalism adds a new computational tool to the standard index notation, namely the soldering isomorphism σ. For example, in order to derive equation (1) from equation (3) with one tangent space, one writes the full connection as

$$\Gamma^{\alpha}{}_{\beta\gamma} = \{^{\alpha}{}_{\beta\gamma}\} + \lambda^{\alpha}{}_{\beta\gamma} \quad , \tag{5}$$

and then computes the mixed covariant derivative:

$$\begin{aligned} \nabla_{\tilde{\beta}} \hat{R}^{\hat{\mu}}{}_{\hat{\nu}\tilde{\gamma}\tilde{\delta}} &= \partial_{\beta} \hat{R}^{\mu}{}_{\nu\gamma\delta} + \Gamma^{\mu}{}_{\kappa\beta} \hat{R}^{\kappa}{}_{\nu\gamma\delta} - \Gamma^{\kappa}{}_{\nu\beta} \hat{R}^{\mu}{}_{\kappa\gamma\delta} \\ &\quad - \{^{\kappa}{}_{\gamma\beta}\} \hat{R}^{\mu}{}_{\nu\kappa\delta} - \{^{\kappa}{}_{\delta\beta}\} \hat{R}^{\mu}{}_{\nu\gamma\kappa} \\ &= \nabla_{\tilde{\beta}} \hat{R}^{\tilde{\mu}}{}_{\tilde{\nu}\tilde{\gamma}\tilde{\delta}} + \lambda^{\mu}{}_{\kappa\beta} \hat{R}^{\kappa}{}_{\nu\gamma\delta} - \lambda^{\kappa}{}_{\nu\beta} \hat{R}^{\mu}{}_{\kappa\gamma\delta} \quad . \end{aligned}$$

On the other hand, in the two tangent space formalism, one writes the curvature as

$$\hat{R}^{\hat{\mu}}{}_{\hat{\nu}\tilde{\gamma}\tilde{\delta}} = (\sigma^{-1})^{\hat{\mu}}{}_{\tilde{\tau}}\, \sigma^{\tilde{\rho}}{}_{\hat{\nu}}\, \hat{R}^{\tilde{\tau}}{}_{\tilde{\rho}\tilde{\gamma}\tilde{\delta}} \quad , \tag{6}$$

writes the covariant derivatives of σ and σ^{-1} as

$$\nabla_{\tilde{\beta}} (\sigma^{-1})^{\hat{\mu}}{}_{\tilde{\tau}} = (\sigma^{-1})^{\hat{\mu}}{}_{\tilde{\kappa}}\, \lambda^{\tilde{\kappa}}{}_{\tilde{\tau}\tilde{\beta}} \; , \tag{7}$$

$$\nabla_{\tilde{\beta}}\, \sigma^{\tilde{\rho}}{}_{\hat{\nu}} = -\, \sigma^{\tilde{\kappa}}{}_{\hat{\nu}}\, \lambda^{\tilde{\rho}}{}_{\tilde{\kappa}\tilde{\beta}}\ , \tag{8}$$

and then computes the mixed covariant derivative by Leibniz's rule:

$$\begin{aligned}
\nabla_{\tilde{\beta}}\, \hat{R}^{\hat{\mu}}{}_{\hat{\nu}\tilde{\gamma}\tilde{\delta}} &= [\nabla_{\tilde{\beta}}(\sigma^{-1})^{\hat{\mu}}{}_{\tilde{\tau}}]\sigma^{\tilde{\rho}}{}_{\hat{\nu}}\, \hat{R}^{\tilde{\tau}}{}_{\tilde{\rho}\tilde{\gamma}\tilde{\delta}} + (\sigma^{-1})^{\hat{\mu}}{}_{\tilde{\tau}}[\nabla_{\tilde{\beta}}\, \sigma^{\tilde{\rho}}{}_{\hat{\nu}}]\hat{R}^{\tilde{\tau}}{}_{\tilde{\rho}\tilde{\gamma}\tilde{\delta}} \\
&\quad +(\sigma^{-1})^{\hat{\mu}}{}_{\tilde{\tau}}\, \sigma^{\tilde{\rho}}{}_{\hat{\nu}}\, \nabla_{\tilde{\beta}}\, \hat{R}^{\tilde{\tau}}{}_{\tilde{\rho}\tilde{\gamma}\tilde{\delta}} \\
&= (\sigma^{-1})^{\hat{\mu}}{}_{\tilde{\tau}}\, \sigma^{\tilde{\rho}}{}_{\hat{\nu}}(\nabla_{\tilde{\beta}}\, \hat{R}^{\tilde{\tau}}{}_{\tilde{\rho}\tilde{\gamma}\tilde{\delta}} + \lambda^{\tilde{\tau}}{}_{\tilde{\kappa}\tilde{\beta}}\, \hat{R}^{\tilde{\kappa}}{}_{\tilde{\rho}\tilde{\gamma}\tilde{\delta}} - \lambda^{\tilde{\kappa}}{}_{\tilde{\rho}\tilde{\beta}}\, \hat{R}^{\tilde{\tau}}{}_{\tilde{\kappa}\tilde{\gamma}\tilde{\delta}}).
\end{aligned}$$

The computation is not any simpler but gives a more intuitive understanding of internalizing and externalizing of indices.

The Cartan differential form notation is specifically designed to handle an independent connection on a single tangent space. For example, if

$$\psi^{\mu}{}_{\nu} = \frac{1}{6}\,\psi^{\mu}{}_{\nu\beta\gamma\delta}\;\theta^{\beta} \wedge \theta^{\gamma} \wedge \theta^{\delta} \quad , \tag{9}$$

is a T^1_1M valued 3-form, then its exterior covariant derivative is the T^1_1M valued 4-form,

$$D\psi^{\mu}{}_{\nu} = d\psi^{\mu}{}_{\nu} + \omega^{\mu}{}_{\kappa} \wedge \psi^{\kappa}{}_{\nu} - \omega^{\kappa}{}_{\nu} \wedge \psi^{\mu}{}_{\kappa} \quad , \tag{10}$$

where d is the exterior derivative and

$$\omega^{\mu}{}_{\kappa} = \Gamma^{\mu}{}_{\kappa\alpha}\,\theta^{\alpha} \quad , \tag{11}$$

is the connection 1-form of the full connection. In terms of mixed covariant derivatives.

$$D\psi^{\mu}{}_{\nu} = \frac{1}{6}\,(\nabla_{\alpha}\psi^{\hat{\mu}}{}_{\hat{\nu}\tilde{\beta}\tilde{\gamma}\tilde{\delta}})\;\theta^{\alpha} \wedge \theta^{\beta} \wedge \theta^{\gamma} \wedge \theta^{\delta} \quad . \tag{12}$$

Notice that the "visible" tensor indices (μ and ν) are corrected with the full connection in both equations (10) and (12), while the "invisible" form indices (β, γ and δ) remain uncorrected in equation (10) and are corrected with the Christoffel connection in equation (12). In fact, in a coordinate basis, the antisymmetrization on α, β, γ and δ cancels all the Christoffel symbols in equation (12). Setting

$$\psi^{\mu}{}_{\nu} = *\,S^{\mu}{}_{\nu} = *(S^{\mu}{}_{\nu\alpha}\;\theta^{\alpha}) = \frac{1}{6}\,S^{\mu}{}_{\nu\alpha}\;\eta^{\alpha}{}_{\beta\gamma\delta}\;\theta^{\beta} \wedge \theta^{\gamma} \wedge \theta^{\delta} \;, \tag{13}$$

equation (12) becomes

$$D * S^{\mu}{}_{\nu} = (\nabla_{\alpha}\; S^{\hat{\mu}}{}_{\hat{\nu}}{}^{\tilde{\alpha}})\,\eta \quad . \tag{14}$$

Here the Christoffel correction occurs in a divergence.

As exemplified by equations (12) and (14), the Cartan notation is most useful in two situations: (a) the differentiating index is antisymmetrized with all the Christoffel corrected indices, and (b) the Christoffel corrected indices are antisymmetrized and then the differentiating index is contracted with one of them. Luckily, the mixed covariant derivatives appearing in the Bianchi identities, conservation laws and most field equations occur in one of these two situations. So the Cartan notation is a useful formalism. However, I know of no simple way in which a mixed covariant derivative, such as $\nabla_\alpha S^{\hat\mu}{}_{\hat\nu\tilde\beta}$, can be expressed in Cartan notation.

So far I have only discussed the Cartan notation in the context of a single tangent space. It can be easily generalized to two tangent spaces in essentially the same way it is generalized to handle gauge symmetries. For example, if

$$\psi^{B}{}_{\hat\mu}{}^{\tilde\nu} = \frac{1}{2}\,\psi^{B}{}_{\hat\mu}{}^{\tilde\nu}{}_{\tilde\beta\tilde\gamma}\,\theta^{\tilde\beta}\wedge\theta^{\tilde\gamma} \quad , \tag{15}$$

is an internal symmetry vector, internal tangent 1-form, external tangent vector valued 2-form, then its exterior covariant derivative is

$$D\psi^{B}{}_{\hat\mu}{}^{\tilde\nu} = d\psi^{B}{}_{\hat\mu}{}^{\tilde\nu} + A^{B}{}_{C}\wedge\psi^{C}{}_{\hat\mu}{}^{\tilde\nu} - \hat\omega^{\hat\kappa}{}_{\hat\mu}\wedge\psi^{B}{}_{\hat\kappa}{}^{\tilde\nu} + \tilde\omega^{\tilde\nu}{}_{\tilde\kappa}\wedge\psi^{B}{}_{\hat\mu}{}^{\tilde\kappa} \ , \tag{16}$$

$$= \frac{1}{2}\,(\nabla_{\tilde\alpha}\,\psi^{B}{}_{\hat\mu}{}^{\tilde\nu}{}_{\tilde\beta\tilde\gamma})\,\theta^{\tilde\alpha}\wedge\theta^{\tilde\beta}\wedge\theta^{\tilde\gamma} \ , \tag{17}$$

where $\hat\omega^{\hat\kappa}{}_{\hat\mu}$ is the internal tangent connection 1-form, $\tilde\omega^{\tilde\kappa}{}_{\tilde\mu}$ is the external tangent connection 1-form, and $A^{B}{}_{C}$ is the internal symmetry connection 1-form written as a matrix in the same representation as ψ. Notice that the visible indices (B, $\hat{\mu}$ and $\tilde{\nu}$) can have any type of corrections, while the invisible indices ($\tilde{\beta}$ and $\tilde{\gamma}$) always have Christoffel corrections.

What happens when an invisible index is made visible? With a single tangent space, this is accomplished with an insertion operator. Thus if

$$\psi^{\mu} = \frac{1}{2} \psi^{\mu}{}_{\beta\gamma} \theta^{\beta} \wedge \theta^{\gamma} , \tag{18}$$

then

$$\iota_{\nu} \psi^{\mu} = \psi^{\mu}{}_{\nu\gamma} \theta^{\gamma} . \tag{19}$$

As a visible index, ν is automatically corrected with a full connection in covariant derivatives. With two tangent spaces, there are two insertion operators. Thus if

$$\psi^{\hat{\mu}} = \frac{1}{2} \psi^{\hat{\mu}}{}_{\tilde{\beta}\tilde{\gamma}} \theta^{\tilde{\beta}} \wedge \theta^{\tilde{\gamma}} , \tag{20}$$

then

$$\iota_{\tilde{\nu}} \psi^{\hat{\mu}} = \psi^{\hat{\mu}}{}_{\tilde{\nu}\tilde{\gamma}} \theta^{\tilde{\gamma}} , \tag{21}$$

and by convention

$$\iota_{\hat{\kappa}} \psi^{\hat{\mu}} = \sigma^{\tilde{\nu}}{}_{\hat{\kappa}} \iota_{\tilde{\nu}} \psi^{\hat{\mu}} = \psi^{\hat{\mu}}{}_{\hat{\kappa}\tilde{\gamma}} \theta^{\tilde{\gamma}} . \tag{22}$$

Each has the corresponding correction in covariant derivatives.

The Cartan notation with two tangent spaces becomes identical in appearance with the Cartan notation with a single tangent space if all visible indices are internal (whether tangent or gauge) and all insertions are made in internal directions.

4. Minimal Coupling: Which Indices are Which?

In this section, I discuss the question, "Which tangent indices are internal and which are external?" In one sense, there is no content in this question, since the soldering isomorphism can be used to convert between internal and external tangent indices. There is somewhat more content when constructing covariant derivatives, since one must decide whether to correct each index with the Christoffel connection or with the full connection. Even then, one can convert between the two types of covariant derivatives by adding correction terms using the defect tensor. The question only really becomes significant when one is constructing the matter Lagrangian (or matter field equations).

The following discussion of minimal coupling continues the discussion of the matter Lagrangian in Section II.1, and the discussion of the full Lagrangian in Section II.5. The procedure for minimally coupling the matter Lagrangian to the gravitational field is analogous to the procedure for minimally coupling the source Lagrangian to the gauge field as specified by equation (II.1.30).

Recall that the full Lagrangian may be decomposed as in Section II.5:

$$L(\psi,g,\theta,\Gamma,A) = L_G(g,\theta,\Gamma) + L_M(\psi,A) + L_I'(\psi,g,\theta,\Gamma,A), \qquad (1)$$

where

$$L_M(\psi,A) = L(\psi,g^{\circ}_{\alpha\beta},\delta^{\alpha}{}_{b},0,A) + \frac{\hbar c}{8\pi L^2}\Lambda, \qquad (2)$$

$$L_G(g,\theta,\Gamma) = L(0,g,\theta,\Gamma,0) - L_C , \qquad (3)$$

$$L(0,g^{\circ}_{\alpha\beta},\delta^{\alpha}{}_{b},0,0) = -\frac{\hbar c}{8\pi L^2}\Lambda + L_C . \qquad (4)$$

Here, $g^{\circ}_{\alpha\beta}$ is any constant value of the metric appropriate to the special relativistic limit with spacetime symmetry group, G_1; the cosmological constant is Λ which satisfies

$$-\frac{\hbar c}{8\pi L^2}\Lambda = L_G(g^{\circ}_{\alpha\beta}, \delta^{\alpha}{}_{b}, 0); \tag{5}$$

and L_C is the constant term in the matter Lagrangian satisfying

$$L_C = L_M(0,0). \tag{6}$$

The matter Lagrangian may then be decomposed as in Section II.1:

$$L_M(\psi,A) = L_A(A) + L_S(\psi) + L_I(\psi,A) + L_C , \tag{7}$$

where

$$L_S(\psi) = L_M(\psi,0) - L_C , \tag{8}$$

$$L_A(A) = L_M(0,A) - L_C . \tag{9}$$

For the purposes of discussing minimal coupling to the gravitational field, I assume that (1) the matter Lagrangian is itself minimally coupled to the gauge fields, and (2) the gauge Lagrangian is minimally constructed. Thus the interacting source Lagrangian may be obtained from the (global gauge theory) source Lagrangian by replacing all partial derivatives, ∂, by gauge covariant derivatives, $\overset{2}{\nabla}\psi$:

$$L_S(\psi,\partial\psi,\ldots,\partial^{(m)}\psi) + L_I(\psi,\partial\psi,\ldots,\partial^{(m)}\psi, A, \partial A,\ldots, \partial^{(n)}A)$$

$$= L_S(\psi, \overset{2}{\nabla}\psi,\ldots, \overset{2}{\nabla}{}^{(m)}\psi). \tag{10}$$

Further, the gauge Lagrangian is only a function of the gauge curvature, F, and its gauge covariant derivatives:

$$L_A(A, \partial A, \ldots, \partial^{(n)}A) = L_A(F, \overset{2}{\nabla}F, \ldots, \overset{2}{\nabla}{}^{(p)}F). \tag{11}$$

Consequently, the total matter Lagrangian can be written as

$$L_M(\psi, \partial\psi, \ldots, \partial^{(m)}\psi, A, \partial A, \ldots, \partial^{(n)}A)$$
$$= L_A(F, \overset{2}{\nabla}F, \ldots, \overset{2}{\nabla}{}^{(p)}F) + L_S(\psi, \overset{2}{\nabla}\psi, \ldots, \overset{2}{\nabla}{}^{(m)}\psi) + L_C. \tag{12}$$

Note that $\overset{2}{\nabla}$ is covariant under gauge transformations of the group, G_2, but not under internal tangent frame transformation of the group, G_1, nor under general external tangent frame transformations.

I first discuss minimal coupling to gravity for a *metric theory* with the spacetime symmetry group, G_1, chosen as O(3,1,R), SL(2,C) or one of their subgroups. In these cases the admissible frames, $e_\alpha{}^a$, are orthonormal and the components of the metric $g_{\alpha\beta}$, and its special relativistic limit, $\overset{\circ}{g}_{\alpha\beta}$, are both the Minkowski metric:

$$g_{\alpha\beta} = \overset{\circ}{g}_{\alpha\beta} = \eta_{\alpha\beta} = \text{diag}(s,-s,-s,-s). \tag{13}$$

Using the usual *one tangent space* formalism, the full Lagrangian, L in (1) is said to be minimally coupled to the gravitational field if the interacting matter Lagrangian, $L_M + L_I'$, may be obtained from the (special relativistic) matter Lagrangian, L_M in (12), by the two step prescription: (i) replace all gauge covariant derivatives, $\overset{2}{\nabla}$, by spacetime and gauge covariant derivatives, ∇; and then (ii) convert all coordinate indices into orthonormal indices by contracting with $e_\alpha{}^a$ or $\theta^\alpha{}_a$. Thus $L_M + L_I'$ is required to be related to L_M by

$$L_M(\psi,A) + L_I'(\psi,\eta,\theta,\Gamma,A)$$
$$= L_A(\theta[F],\theta[\nabla F],\ldots, \theta[\nabla^{(p)}F])$$
$$+ L_S(\theta[\psi], \theta[\nabla\psi],\ldots, \theta[\nabla^{(m)}\psi])$$
$$+ L_C . \tag{14}$$

Here, if T is any tensor with some coordinate and some orthonormal indices, $\theta[T]$ denotes the same tensor with all orthonormal indices. In the *two tangent space* formalism the prescription for minimal coupling is the same except that all coordinate indices are regarded as external, all orthonormal indices are regarded as internal, and the frames $e_\alpha{}^a$ and $\theta^\alpha{}_a$ are replaced by the soldering isomorphism $\sigma_\alpha{}^a$ and its inverse $(\sigma^{-1})^\alpha{}_a$. Of course, for a metric theory it does not matter whether one converts the external indices into internal indices before or after taking covariant derivatives. So step (ii) of the prescription is equivalent to: (ii') write everything in (internal) orthonormal indices in the first place. Also in a metric theory the definition of F in terms of A can be written using either partial derivatives or spacetime covariant derivatives since the connection has no torsion. So step (i) of the prescription is equivalent to: (i') replace all partial derivatives, ∂, by spacetime covariant derivatives, $\overset{1}{\nabla}$. This automatically converts gauge covariant derivatives, $\overset{2}{\nabla}$, into spacetime and gauge covariant derivatives, ∇. I prefer step (i) and (ii) rather than (i') and (ii') because (i) and (ii) generalize to the non-metric theories while (i') and (ii') do not.

I next discuss minimal coupling for a *metric-Cartan connection theory*. Thus the spacetime symmetry group, G_1, is still O(3,1,R), SL(2,C) or one of their subgroups; the admissible frames, $e_\alpha{}^a$, are still orthonormal; and the components of the metric, $g_{\alpha\beta}$, and its special relativistic limit, $\overset{\circ}{g}_{\alpha\beta}$, are still the Minkowski metric as in (13). In the *two tangent space* formalism, the full Lagrangian, L in (1), is said to be minimally coupled to the gravitational field if the interacting matter Lagrangian, $L_M + L_I'$, may be obtained from the (special relativistic) matter Lagrangian, L_M in (12), by the two step prescription: (i) replace all gauge covariant derivatives, $\overset{2}{\nabla}$, by internal spacetime, external spacetime and gauge covariant derivatives, ∇; and then (ii) convert all external tangent indices into internal tangent indices by contracting with $\sigma_\alpha{}^a$ or $(\sigma^{-1})^\alpha{}_a$. Thus $L_M + L_I'$ is required to be related to L_M by

$$\begin{aligned} L_M(\psi,A) &+ L_I'(\psi,\eta,\sigma^{-1},\Gamma,A) \\ &= L_A(\sigma^{-1}[F],\ \sigma^{-1}[\nabla F],\ldots,\sigma^{-1}[\nabla^{(p)}F]) \\ &\quad + L_S(\sigma^{-1}[\psi],\sigma^{-1}[\nabla\psi],\ldots,\sigma^{-1}[\nabla^{(m)}\psi]) \\ &\quad + L_C\ , \end{aligned} \tag{15}$$

where $\sigma^{-1}[T]$ is the tensor T with all internal tangent indices. Notice that this prescription is identical to that for a metric theory, but it is now crucial that (1) any external tangent indices are converted to internal tangent indices *after* performing covariant derivatives and (2) the partial derivatives in F are not converted into Cartan covariant derivatives since that would destroy the gauge covariance of F.

This brings us back to the original question in this section, "Which tangent indices are internal or external?" Before performing the covariant derivatives in (15) one must answer this question for each of the tangent indices on ψ, F and ∇. It is obvious that there is no absolute answer to this question, but I regard the following answer as standard: All differentiating indices are external. This includes the differentiating indices on ∇, A and F. (Thus one could say that the partial derivatives in the definition of F are replaced by Christoffel covariant derivatives.) All tangent indices on the source field, ψ, are internal.

This standard choice between internal and external tangent indices agrees with that made by Hehl, von der Heyde, Kerlick and Nester [1976] in their review of the ECSK theory. It also agrees with the choice made by the Cartan differential form notation in that the visible indices correspond to the internal tangent indices and are corrected using the full connection while the invisible indices correspond to the external tangent indices and are corrected using the Christoffel connection.

An important non-standard source field is the spin 3/2 field, $\psi^A{}_a$, of supergravity for which the spinor index, A, is regarded as internal while the 1-form index, a, is regarded as external. (Of course, it is possible that $\psi^A{}_a$ should not be interpreted as a source field but rather as a piece of the spacetime connection, Γ, with the spacetime symmetry group, G_1, enlarged to include the supersymmetry transformations.) In the Cartan form notation, the spin 3/2 field is described as the spinor valued 1-form, $\psi^A = \psi^A{}_a\, dx^a$. This brings up an important advantage of the two tangent space formalism over the Cartan form notation. In the two tangent space formalism it is possible to consider non-standard source fields in which the external indices are not antisymmetrized as they must be in a differential form.

See also the paper by Hojman, Rosenbaum, Ryan and Shepley [1978] in which they consider a non-standard minimal coupling of the gauge Lagrangian.

I finally discuss minimal coupling for a metric-connection theory in which the spacetime symmetry group, G_1, is larger than O(3,1,R) or SL(2,C). Then the admissible frames, $e_\alpha{}^a$, need not be orthonormal, and the components of the metric, $g_{\alpha\beta}$, need not be the Minkowski metric. Further, $g_{\alpha\beta}$ need not even be constant. To obtain the special relativistic limit, one chooses a constant matrix, $\overset{\circ}{g}_{\alpha\beta}$, for each patch which are related by a constant G_1 transformation on the overlap of each pair of patches.

Notice that equations (1,2, 4-12), depend implicitly on the choice of the constant matrix, $\overset{\circ}{g}_{\alpha\beta}$. I now make that dependence explicit, rewriting equations (1) and (12) as

$$L(\psi,g,\sigma^{-1},\Gamma,A) = L_G(g,\sigma^{-1},\Gamma) + L_M(\psi,A,\overset{\circ}{g}_{\alpha\beta}) + L_I'(\psi,g,\sigma^{-1},\Gamma,A,\overset{\circ}{g}_{\alpha\beta}), \tag{16}$$

$$L_M(\psi,\partial\psi,\ldots,\partial^{(m)}\psi, A,\partial A,\ldots,\partial^{(n)}A,\overset{\circ}{g}_{\alpha\beta})$$

$$= L_A(F,\overset{2}{\nabla}F,\ldots,\overset{2}{\nabla}{}^{(p)}F,\overset{\circ}{g}_{\alpha\beta}) + L_S(\psi,\overset{2}{\nabla}\psi,\ldots,\overset{2}{\nabla}{}^{(m)}\psi,\overset{\circ}{g}_{\alpha\beta}) + L_C. \tag{17}$$

In the *two tangent space* formalism, the full Lagrangian, L in (16), is said to be minimally coupled to the gravitational field if the interacting matter Lagrangian, $L_M + L_I'$, may be obtained from the (special relativistic) matter Lagrangian, L_M in (17), by the three step prescription: (i) replace all gauge covariant derivatives, $\overset{2}{\nabla}$, by internal spacetime, external spacetime and gauge covariant derivatives, ∇; then (ii) convert all external tangent indices into internal tangent indices by contracting with $\sigma_\alpha{}^a$ or $(\sigma^{-1})^\alpha{}_a$; and (iii) replace the constant matrix, $\overset{\circ}{g}_{\alpha\beta}$, by the nonconstant metric, $g_{\alpha\beta}$. Thus $L_M + L_I'$ is required to be related to L_M by

$$L_M(\psi, A, g^\circ_{\alpha\beta}) + L'_I(\psi, g, \sigma^{-1}, \Gamma, A, g^\circ_{\alpha\beta})$$

$$= L_A(\sigma^{-1}[F],\ \sigma^{-1}[\nabla F], \ldots,\ \sigma^{-1}[\nabla^{(p)}F], g_{\alpha\beta})$$

$$+ L_S(\sigma^{-1}[\psi],\ \sigma^{-1}[\nabla\psi], \ldots,\ \sigma^{-1}[\nabla^{(m)}\psi],\ g_{\alpha\beta})$$

$$+ L_C. \tag{18}$$

The standard choice between internal and external tangent indices is the same as that made in the case of a metric-Cartan connection theory.

5. Noether's Theorem and Conservation Laws

a. General Formalism

In this section, I discuss Noether's theorem in the context of the two tangent space formalism. The fact that the matter Lagrangian, L_M, is a scalar under coordinate transformations leads to the conservation of energy-momentum. The fact that L_M is a scalar under internal tangent frame transformations leads to the conservation of angular-momentum and possibly dilation current and hypermomentum. Unlike previous sections, I here use Latin indices to denote a coordinate basis on the external tangent bundle, and Greek indices to denote an arbitrary G-compatible basis on the internal tangent bundle. The carets over Greek indices and tildes over Latin indices are not written.

In this section, I ignore any gauge fields and assume the interacting matter Lagrangian,

$$L_M = L_M \left(\psi^{(X)},\ \partial_a \psi^{(X)},\ \Gamma^\alpha{}_{\beta a},\ (\sigma^{-1})^\alpha{}_a,\ g_{\alpha\beta} \right) , \qquad (1)$$

is a scalar function of only

(a) the internal components of the source fields, $\psi^{(X)}$, (Here (X) denotes any collection of internal tangent indices, spinor indices, gauge indices, any/or indices to count off different fields, which transform according to some representation, R_ψ, of the group, G.)

(b) their coordinate partial derivatives, $\partial_a \psi^{(X)}$,

(c) the internal tangent connection coefficients, $\Gamma^\alpha{}_{\beta a}$, (to construct covariant derivatives in coordinate directions,)

(d) the components of the inverse soldering isomorphism, $(\sigma^{-1})^\alpha{}_a$, (to convert external coordinate indices into internal indices,) and

(e) the internal components of the metric, $g_{\alpha\beta}$, (to contract indices).

Notice that other variables may be constructed algebraically from these. In particular, the soldering isomorphism, $\sigma^a{}_\alpha$, and the inverse metric, $g^{\alpha\beta}$, are the solutions of the linear equations,

$$\sigma^a{}_\alpha \, (\sigma^{-1})^\alpha{}_b = \delta^a{}_b \, , \tag{2}$$

$$g^{\alpha\beta} \, g_{\beta\gamma} = \delta^\alpha{}_\gamma \, , \tag{3}$$

the external components of the metric are

$$g_{ab} = g_{\alpha\beta} \, (\sigma^{-1})^\alpha{}_a \, (\sigma^{-1})^\beta{}_b \, , \tag{4}$$

and the determinant of the external metric is

$$\tilde{g} = \det g_{ab} = (\det g_{\alpha\beta}) \, [\det (\sigma^{-1})^\alpha{}_a]^2 \, . \tag{5}$$

The matter action may be written in several forms,

$$S_M = \int L_M \, \eta = \int L_M \sqrt{-\tilde{g}} \, d^4x = \int \mathcal{L}_M \, d^4x \, , \tag{6}$$

where the volume element is

$$\eta = \sqrt{-\tilde{g}} \, d^4x \, , \tag{7}$$

and the matter Lagrangian density is

$$\mathcal{L}_M = L_M \sqrt{-\tilde{g}} \, , \tag{8}$$

$$= \mathcal{L}_M\left(\psi^{(X)}, \, \partial_\alpha \psi^{(X)}, \, \Gamma^\alpha{}_{\beta a}, \, (\sigma^{-1})^\alpha{}_a, \, g_{\alpha\beta}\right) \, . \tag{9}$$

The Lagrangian density may be varied in two ways. First, varying equation (8), using the Leibniz rule, yields

$$\mathcal{L}_M = \sqrt{-\tilde{g}} \, \delta L_M + L_M \, \delta\sqrt{-\tilde{g}} = \sqrt{-\tilde{g}} \, [\delta L_M + \tfrac{1}{2} L_M \, g^{ab} \, \delta g_{ab}] \, . \tag{10}$$

Second, varying equation (9), using the chain rule, yields

$$\delta\mathcal{L}_M = \frac{\partial\mathcal{L}_M}{\partial g_{\alpha\beta}}\,\delta g_{\alpha\beta} + \frac{\partial\mathcal{L}_M}{\partial(\sigma^{-1})^\alpha{}_a}\,\delta(\sigma^{-1})^\alpha{}_a + \frac{\partial\mathcal{L}_M}{\partial\Gamma^\alpha{}_{\beta a}}\,\delta\Gamma^\alpha{}_{\beta\alpha}$$

$$+ \frac{\partial\mathcal{L}_M}{\partial\psi^{(X)}}\,\delta\psi^{(X)} + \frac{\partial\mathcal{L}_M}{\partial\partial_a\psi^{(X)}}\,\delta\partial_a\psi^{(X)}. \tag{11}$$

Since variations commute with coordinate partial derivatives, equation (11) may be rewritten as

$$\delta\mathcal{L}_M = \frac{\partial\mathcal{L}_M}{\partial g_{\alpha\beta}}\,\delta g_{\alpha\beta} + \frac{\partial\mathcal{L}_M}{\partial(\sigma^{-1})^\alpha{}_a}\,\delta(\sigma^{-1})^\alpha{}_a + \frac{\partial\mathcal{L}_M}{\partial\Gamma^\alpha{}_{\beta a}}\,\delta\Gamma^\alpha{}_{\beta a}$$

$$+ \left(\frac{\partial\mathcal{L}_M}{\partial\psi^{(X)}} - \partial_a\frac{\partial\mathcal{L}_M}{\partial\partial_a\psi^{(X)}}\right)\delta\psi^{(X)} + \partial_a\left(\frac{\partial\mathcal{L}_M}{\partial\partial_a\psi^{(X)}}\,\delta\psi^{(X)}\right). \tag{12}$$

I now introduce the <u>symmetric energy-momentum tensor</u>, $T^{\alpha\beta}$, the <u>canonical energy-momentum tensor</u>, $t_\alpha{}^a$, the <u>canonical spin tensor</u>, $S^\beta{}_\alpha{}^a$, and the matter <u>Euler-Lagrange tensor</u>, $L_{(X)}$, which are defined by the formulas,

$$-s\sqrt{-\tilde{g}}\,\frac{1}{2}\,T^{\alpha\beta} = \frac{\delta\mathcal{L}_M}{\delta g_{\alpha\beta}} = \frac{\partial\mathcal{L}_M}{\partial g_{\alpha\beta}}, \tag{13}$$

$$-s\sqrt{-\tilde{g}}\,t_\alpha{}^a = \frac{\delta\mathcal{L}_M}{\delta(\sigma^{-1})^\alpha{}_a} = \frac{\partial\mathcal{L}_M}{\partial(\sigma^{-1})^\alpha{}_a}, \tag{14}$$

$$-s\sqrt{-\tilde{g}}\,\frac{1}{2}\,S^\beta{}_\alpha{}^a = \frac{\delta\mathcal{L}_M}{\delta\Gamma^\alpha{}_{\beta a}} = \frac{\partial\mathcal{L}_M}{\partial\Gamma^\alpha{}_{\beta a}}, \tag{15}$$

$$\sqrt{-\tilde{g}}\,L_{(X)} = \frac{\delta\mathcal{L}_M}{\delta\psi^{(X)}} = \frac{\partial\mathcal{L}_M}{\partial\psi^{(X)}} - \partial_a\frac{\partial\mathcal{L}_M}{\partial\partial_a\psi^{(X)}}. \tag{16}$$

Throughout the rest of this section, I use the spacelike convention for the signature of the metric. Thus, $s = -1$.

Also recall an identity for Christoffel covariant derivatives:

$$\partial_a(\sqrt{-\tilde{g}}\ \ V^a) = \sqrt{-\tilde{g}}\ \nabla_a V^a \ . \tag{17}$$

Using equation (8) and

$$V^a = \frac{\partial L_M}{\partial\partial_a\psi^{(X)}}\ \ \delta\psi^{(X)}, \tag{18}$$

equation (17) becomes

$$\partial_a\left(\frac{\partial\mathcal{L}_M}{\partial\partial_a\psi^{(X)}}\ \ \delta\psi^{(X)}\right) = \sqrt{-\tilde{g}}\ \ \nabla_a\left(\frac{\partial L_M}{\partial\partial_a\psi^{(X)}}\ \ \delta\psi^{(X)}\right) . \tag{19}$$

Note: For coordinate transformations, $\delta\psi^{(X)} = -\ \varepsilon^b\partial_b\psi^{(X)}$ is not a tensor under frame transformations. Consequently, $\nabla_a(\frac{\partial L_M}{\partial\partial_a\psi^{(X)}}\ \delta\psi^{(X)})$ is not really a covariant derivative. Instead it is a shorthand notation for

$$\nabla_a\left(\frac{\partial L_M}{\partial\partial_a\psi^{(X)}}\ \ \delta\psi^{(X)}\right) = \partial_a\left(\frac{\partial L_M}{\partial\partial_a\psi^{(X)}}\ \ \delta\psi^{(X)}\right) + \{^a_{\ ba}\}\frac{\partial L_M}{\partial\partial_b\psi^{(X)}}\ \ \delta\psi^{(X)} . \tag{20}$$

Using (13), (14), (15), (16), and (19), equation (12) becomes

$$\delta\mathcal{L}_M = \sqrt{-\tilde{g}}\ \left[\frac{1}{2}\ T^{\alpha\beta}\delta g_{\alpha\beta} + t_\alpha^{\ a}\delta(\sigma^{-1})^\alpha_{\ a} + \frac{1}{2}\ S^\beta{}_\alpha{}^a\delta\Gamma^\alpha{}_{\beta a}\right.$$
$$\left. + L_{(X)}\delta\psi^{(X)} + \ \nabla_a\left(\frac{\partial L_M}{\partial\partial_a\psi^{(X)}}\ \ \delta\psi^{(X)}\right)\right] . \tag{21}$$

Equating expressions (10) and (21) for $\delta\mathcal{L}_M$ yields,

$$\delta L_M + \frac{1}{2}\ L_M\ g^{ab}\delta g_{ab} = \frac{1}{2}\ T^{\alpha\beta}\ \delta g_{\alpha\beta} + t_\alpha^{\ a}\ \delta(\sigma^{-1})^\alpha_{\ a} + \frac{1}{2}\ S^\beta{}_\alpha{}^a\ \delta\Gamma^\alpha{}_{\beta a}$$
$$+ L_{(X)}\ \delta\psi^{(X)} + \nabla_a\left(\frac{\partial L_M}{\partial\partial_a\psi^{(X)}}\ \delta\psi^{(X)}\right) . \tag{22}$$

In the following subsections, I specialize equation (22) to coordinate variations and certain classes of frame variations. However, I first make two comments.

(1) The derivation so far, and most of that which follows, does not require that L_M is actually the matter Lagrangian. It merely requires that L_M is a scalar function of the variables shown in equation (1). If L_M is only a term in the matter Lagrangian, then $T^{\alpha\beta}$, $t_\alpha{}^a$, $S^\beta{}_\alpha{}^a$ and $L_{(X)}$ are the corresponding contributions to the symmetric energy-momentum tensor, the canonical energy-momentum tensor, the canonical spin tensor and the matter Euler-Lagrange tensor respectively. If L_M is the whole matter Lagrangian, then the Euler-Lagrange equations for the matter fields are

$$L_{(X)} = 0. \tag{23}$$

However, I will continue to carry the $L_{(X)}$ terms in order to keep the derivation general.

(2) In definitions (13) through (16) it is important to specify which variables are held fixed during the variations, or equivalently, on which variables the Lagrangian depends. Definitions (13) through (16) assume the functional dependence shown in equation (1). I here discuss three other choices. In each example, $\mathcal{L}'_M$ denotes the matter Lagrangian density regarded as a function of the new variables.

(a) If one regards the external metric, g_{ab}, as the independent variable instead of the internal metric, $g_{\alpha\beta} = g_{ab}\,\sigma^a{}_\alpha\,\sigma^b{}_\beta$, then

$$\frac{\delta \mathcal{L}'_M}{\delta g_{ab}} = \sqrt{-\tilde{g}}\,\frac{1}{2}\,T^{\alpha\beta}\,\sigma^a{}_\alpha\,\sigma^b{}_\beta, \tag{24}$$

but

$$\frac{\delta \mathcal{L}'_M}{\delta (\sigma^{-1})^\alpha{}_a} = \sqrt{-\tilde{g}}\,(t_\alpha{}^a - T_\alpha{}^\beta\,\sigma^a{}_\beta). \tag{25}$$

(b) If one considers the defect tensor, $\lambda^\alpha{}_{\beta\gamma}$, as the independent variable in place of the connection,

$$\Gamma^\alpha{}_{\beta a} = (\sigma^{-1})^\gamma{}_a\,\lambda^\alpha{}_{\beta\gamma} + (\sigma^{-1})^\alpha{}_b\,(\,\partial_a\sigma^b{}_\beta + \{^b{}_{ac}\}\sigma^c{}_\beta), \tag{26}$$

then

$$\frac{\delta \mathcal{L}'_M}{\delta \lambda^\alpha{}_{\beta\gamma}} = \sqrt{-\tilde{g}}\,\frac{1}{2}\,S^\beta{}_\alpha{}^a\,(\sigma^{-1})^\gamma{}_a\,, \tag{27}$$

but $\delta\mathcal{L}'_M/\delta g_{\alpha\beta}$ and $\delta\mathcal{L}'_M/\delta(\sigma^{-1})^\alpha{}_a$ are significantly different from $\delta\mathcal{L}_M/\delta g_{\alpha\beta}$ and $\delta\mathcal{L}_M/\delta(\sigma^{-1})^\alpha{}_a$ because the Lagrangian is now a function of $\partial_a g_{\alpha\beta}$ and $\partial_b(\sigma^{-1})^\alpha{}_a$. In particular

$$\frac{\delta \mathcal{L}'_M}{\delta g_{\alpha\beta}} = \frac{1}{2}\sqrt{-\tilde{g}}\,[T^{\alpha\beta} - \frac{1}{2}\,\sigma^{(\alpha}{}_a\,\sigma^{\beta)}{}_b\nabla_c(S^{abc} + S^{cab} - S^{acb})], \tag{28}$$

appears on the right hand side of the Einstein equation when the metric and defect are considered as the independent variables. The quantity in square brackets in (28) is called the <u>metric energy-momentum tensor,</u> $\tilde{T}^{\alpha\beta}$.

(c) A change of variables can even produce non-tensorial quantities. For example, if $\Gamma^\alpha{}_{\beta\gamma}$ is used in place of $\Gamma^\alpha{}_{\beta a} = (\sigma^{-1})^\gamma{}_a\,\Gamma^\alpha{}_{\beta\gamma}$, then

$$\frac{\delta \mathcal{L}'_M}{\delta \Gamma^\alpha{}_{\beta\gamma}} = \sqrt{-\tilde{g}}\, \frac{1}{2}\, S^\beta{}_\alpha{}^a\, (\sigma^{-1})^\gamma{}_a \quad , \tag{29}$$

but

$$\frac{\delta \mathcal{L}'_M}{\delta (\sigma^{-1})^\alpha{}_a} = \sqrt{-\tilde{g}}\, (t_\alpha{}^a + \frac{1}{2}\, S^\beta{}_\gamma{}^a\, \Gamma^\gamma{}_{\beta\alpha}\,). \tag{30}$$

The quantities $t_\alpha{}^a$, $S^\beta{}_\gamma{}^a$ and $(-\tilde{g})^{-1/2}\ \delta\mathcal{L}'_M/\delta(\sigma^{-1})^\alpha{}_a$ cannot all be tensors. I don't fully understand why, but from experience, $t_\alpha{}^a$ and $S^\beta{}_\gamma{}^a$ turn out to be tensorial. I consider this one of the best reasons for using the mixed components of the connection, $\Gamma^\alpha{}_{\beta a}$, rather than the frame components, $\Gamma^\alpha{}_{\beta\gamma}$.

I now go back to considering the variables shown in equation (1).

b. Conservation Laws with a $GL_\circ(4,R)$-Internal Tangent Frame Bundle

The $GL_\circ(4,R)$ - internal tangent frame bundle, $GL_\circ(M)$, consists of all frames at all points of spacetime. Its connection is permitted to be completely general, non-metric-compatible and/or non-torsion-free. The matter fields, $\psi^{(X)}$, cannot have any spinor indices because there are no spinor representations of $GL_\circ(4,R)$. Consequently, I consider such a theory undesirable, but I compute its conservation laws for completeness. The results are also applicable to $GL(4,R)$ which has the same Lie algebra as $GL_\circ(4,R)$.

Substitute into equation (22), the $GL_\circ(4,R)$ - internal tangent frame variations of L_M, g_{ab}, $g_{\alpha\beta}$, $(\sigma^{-1})^\alpha{}_a$, $\Gamma^\alpha{}_{\beta a}$, and $\psi^{(X)}$ as listed in Table II.10:

$$0 = \frac{1}{2} T^{\alpha\beta}[-\lambda^\gamma{}_\alpha g_{\gamma\beta} - \lambda^\gamma{}_\beta\, g_{\alpha\gamma}] + t_\alpha{}^a[\lambda^\alpha{}_\beta(\sigma^{-1})^\beta{}_a] + \frac{1}{2} S^\beta{}_\alpha{}^a[-\nabla_a\lambda^\alpha{}_\beta]$$
$$+ L_{(X)}\left[\lambda^\alpha{}_\beta\, R_\psi(E^\beta{}_\alpha)^{(X)}{}_{(Y)}\, \psi^{(Y)}\right] + \nabla_a\left[\frac{\partial L_M}{\partial\partial_a\psi^{(X)}}\;\lambda^\alpha{}_\beta R_\psi(E^\beta{}_\alpha)^{(X)}{}_{(Y)}\psi^{(Y)}\right], \tag{31}$$

or

$$0 = \lambda^\alpha{}_\beta\left[-T_\alpha{}^\beta + t_\alpha{}^\beta + L_{(X)}\, R_\psi(E^\beta{}_\alpha)^{(X)}{}_{(Y)}\,\psi^{(Y)} + \nabla_a\left\{\frac{\partial L_M}{\partial\partial_a\psi^{(X)}}\, R_\psi(E^\beta{}_\alpha)^{(X)}{}_{(Y)}\,\psi^{(Y)}\right\}\right]$$
$$+ \nabla_a\lambda^\alpha{}_\beta\left[-\frac{1}{2} S^\beta{}_\alpha{}^a + \frac{\partial L_M}{\partial\partial_a\psi^{(X)}}\, R_\psi(E^\beta{}_\alpha)^{(X)}{}_{(Y)}\,\psi^{(Y)}\right]. \tag{32}$$

Since this equation must be true at all points for all choices of $\lambda^{\alpha}{}_{\beta}$, the coefficients of $\lambda^{\alpha}{}_{\beta}$ and $\nabla_a \lambda^{\alpha}{}_{\beta}$ may be separately equated to zero:

$$T_{\alpha}{}^{\beta} = t_{\alpha}{}^{\beta} + L_{(X)}\, R_{\psi}(E^{\beta}{}_{\alpha})^{(X)}{}_{(Y)}\, \psi^{(Y)} + \nabla_a \left[\frac{\partial L_M}{\partial \partial_a \psi^{(X)}} R_{\psi}(E^{\beta}{}_{\alpha})^{(X)}{}_{(Y)}\, \psi^{(Y)} \right] , \tag{33}$$

$$S^{\beta}{}_{\alpha}{}^{a} = 2\, \frac{\partial L_M}{\partial \partial_a \psi^{(X)}} R_{\psi}(E^{\beta}{}_{\alpha})^{(X)}{}_{(Y)}\, \psi^{(Y)} . \tag{34}$$

Equation (34) is an alternate definition of the canonical spin tensor. It may be compared with the original definition, equation (15) or equivalently,

$$S^{\beta}{}_{\alpha}{}^{a} = 2\, \frac{\partial L_M}{\partial \Gamma^{\alpha}{}_{\beta a}} . \tag{35}$$

Equating (34) and (35) yields,

$$\frac{\partial L_M}{\partial \Gamma^{\alpha}{}_{\beta a}} = \frac{\partial L_M}{\partial \partial_a \psi^{(X)}} R_{\psi}(E^{\beta}{}_{\alpha})^{(X)}{}_{(Y)}\, \psi^{(Y)} . \tag{36}$$

Recall from equation (1) that I did not originally assume that the matter Lagrangian, L_M, was minimally coupled. However, equation (36) permits one to prove that L_M is in fact minimally coupled; i.e. L_M depends on $\partial_a \psi^{(X)}$ and on $\Gamma^{\alpha}{}_{\beta a}$ only through the combination,

$$\nabla_a \psi^{(X)} = \partial_a \psi^{(X)} + \Gamma^{\alpha}{}_{\beta a}\, R_{\psi}(E^{\beta}{}_{\alpha})^{(X)}{}_{(Y)}\, \psi^{(Y)} , \tag{37}$$

which is the covariant derivative.

Proof: Make a change of variables in L_M from $\partial_a \psi^{(X)}$ and $\Gamma^{\alpha}{}_{\beta a}$ to $\nabla_a \psi^{(X)}$ and $\Gamma^{\alpha}{}_{\beta a}$; i.e. define a new function,

$$L'_M\left(\psi^{(X)},\ \nabla_a\psi^{(X)},\ \Gamma^\alpha{}_{\beta a}, (\sigma^{-1})^\alpha{}_a,\ g_{\alpha\beta}\right)$$

$$= L_M\left(\psi^{(X)},\ \nabla_a\psi^{(X)} - \Gamma^\alpha{}_{\beta a}\ R_\psi(E^\beta{}_\alpha)^{(X)}{}_{(Y)}\ \psi^{(Y)}\ ,\right.$$

$$\left.\Gamma^\alpha{}_{\beta a},\ (\sigma^{-1})^\alpha{}_a,\ g_{\alpha\beta}\right) . \tag{38}$$

Use chain rule and equation (36) to compute

$$\frac{\partial L'_M}{\partial\Gamma^\alpha{}_{\beta a}} = \frac{\partial L_M}{\partial\partial_a\psi^{(X)}}\left[-R_\psi(E^\beta{}_\alpha)^{(X)}{}_{(Y)}\ \psi^{(Y)}\right] + \frac{\partial L_M}{\partial\Gamma^\alpha{}_{\beta a}} = 0\ . \tag{39}$$

Consequently, L'_M is independent of $\Gamma^\alpha{}_{\beta a}$ and

$$L_M\left(\psi^{(X)},\ \partial_a\psi^{(X)},\ \Gamma^\alpha{}_{\beta a},\ (\sigma^{-1})^\alpha{}_a,\ g_{\alpha\beta}\right)$$

$$= L'_M\left(\psi^{(X)},\ \nabla_a\psi^{(X)},\ (\sigma^{-1})^\alpha{}_a,\ g_{\alpha\beta}\right)\ , \tag{40}$$

is minimally coupled. Notice again that this proof does not require that L_M is actually the matter Lagrangian. It merely requires that L_M is a scalar function of the variables shown in equation (1).

The canonical spin tensor, equation (34), may be divided into

(1) an antisymmetric part, called the canonical spin (or intrinsic) angular momentum tensor,

$$S_{[\beta\alpha]}{}^a = \frac{\partial L_M}{\partial\partial_a\psi^{(X)}}\ R_\psi(\sigma_{\beta\alpha})^{(X)}{}_{(Y)}\ \psi^{(Y)}\ , \tag{41}$$

where $\sigma_{\beta\alpha} = E_{\beta\alpha} - E_{\alpha\beta}$; (Note that equation (41) is the usual definition of the canonical spin tensor used in particle physics.) and

(2) a symmetric part, which I will call the canonical intrinsic (or spin) hypermomentum tensor,

$$S_{(\beta\alpha)}{}^{a} = \frac{\partial L_M}{\partial \partial_a \psi^{(X)}} \; R_\psi(\tau_{\beta\alpha})^{(X)}{}_{(Y)} \; \psi^{(Y)} \; , \qquad (42)$$

where $\tau_{\beta\alpha} = E_{\beta\alpha} + E_{\alpha\beta}$.

The intrinsic hypermomentum tensor may be further divided into

(1) a trace part, called the <u>canonical intrinsic</u> (or <u>spin</u>) <u>dilation current</u>,

$$S^{\alpha}{}_{\alpha}{}^{a} = 2 \frac{\partial L_M}{\partial \partial_a \psi^{(X)}} \; R_\psi(\mathbb{1})^{(X)}{}_{(Y)} \; \psi^{(Y)} \; , \qquad (43)$$

where $\mathbb{1} = E^{\alpha}{}_{\alpha}$; and

(2) a tracefree part, called the <u>tracefree canonical intrinsic</u> (or <u>spin</u>) <u>hypermomentum tensor</u>,

$$S_{(\beta\alpha)}{}^{a} - \frac{1}{4} g_{\alpha\beta} \, S^{\gamma}{}_{\gamma}{}^{a} = \frac{\partial L_M}{\partial \partial_a \psi^{(X)}} \; R_\psi(\tau_{\beta\alpha} - \frac{1}{2} g_{\beta\alpha} \mathbb{1})^{(X)}{}_{(Y)} \; \psi^{(Y)} . \qquad (44)$$

The canonical spin tensor, equation (34), may be substituted into equation (33), yielding,

$$T_{\alpha}{}^{\beta} = \frac{1}{2} \; \nabla_a S^{\beta}{}_{\alpha}{}^{a} + t_{\alpha}{}^{\beta} + L_{(X)} \; R_\psi(E^{\beta}{}_{\alpha})^{(X)}{}_{(Y)} \; \psi^{(Y)} \; . \qquad (45)$$

The antisymmetric part is

$$0 = \frac{1}{2} \; (g_{\beta\gamma} \; \nabla_a S^{\gamma}{}_{\alpha}{}^{a} - g_{\alpha\gamma} \; \nabla_a S^{\gamma}{}_{\beta}{}^{a}) + 2 \; t_{[\alpha\beta]} + L_{(X)} \; R_\psi(\sigma_{\beta\alpha})^{(X)}{}_{(Y)} \; \psi^{(Y)} \; , \qquad (46)$$

or

$$0 = \nabla_a S_{[\beta\alpha]}{}^{a} + 2 \; t_{[\alpha\beta]} + (\nabla_a g_{\gamma[\alpha}) \; S^{\gamma}{}_{\beta]}{}^{a} + L_{(X)} \; R_\psi(\sigma_{\beta\alpha})^{(X)}{}_{(Y)} \; \psi^{(Y)} \; . \qquad (47)$$

Notice that I have not set $\nabla g = 0$, since I am investigating the $GL_\circ(4,R)$-internal tangent frame bundle with a $GL_\circ(4,R)$-connection. When the matter field equations, $L_{(X)} = 0$, are satisfied, (See the discussion preceding equation (23).) equation (47) becomes the conservation law for angular momentum,

$$0 = \nabla_a S_{[\beta\alpha]}{}^{a} + 2\ t_{[\alpha\beta]} + (\nabla_a g_{\gamma[\alpha})S^{\gamma}{}_{\beta]}{}^{a}. \qquad (48)$$

The symmetric part of equation (45),

$$2\ T_{\alpha\beta} = \frac{1}{2}(g_{\beta\gamma}\ \nabla_a S^{\gamma}{}_{\alpha}{}^{a} + g_{\alpha\gamma}\ \nabla_a S^{\gamma}{}_{\beta}{}^{a})$$

$$+ t_{\alpha\beta} + t_{\beta\alpha} + L_{(X)} R_{\psi}(\tau_{\beta\alpha})^{(X)}{}_{(Y)}\ \psi^{(Y)}, \qquad (49)$$

or

$$2\ T_{\alpha\beta} = \nabla_a S_{(\beta\alpha)}{}^{a} + 2\ t_{(\alpha\beta)} - (\nabla_a g_{\gamma(\alpha})\ S^{\gamma}{}_{\beta)}{}^{a}$$

$$+ L_{(X)}\ R_{\psi}(\tau_{\beta\alpha})^{(X)}{}_{(Y)}\ \psi^{(Y)}, \qquad (50)$$

may be regarded as an alternate definition of the symmetric energy-momentum tensor, $T_{\alpha\beta}$, which generalizes the Belinfante-Rosenfeld symmetrization procedure. When the matter field equations, $L_{(X)} = 0$, are satisfied, equation (50) may be regarded as the conservation law for hypermomentum,

$$0 = \nabla_a S_{(\beta\alpha)}{}^{a} + 2\ t_{(\alpha\beta)} - 2\ T_{\alpha\beta} - (\nabla_a g_{\gamma(\alpha})S^{\gamma}{}_{\beta)}{}^{a}. \qquad (51)$$

In particular the trace of equation (49) is

$$2\ T_{\alpha}{}^{\alpha} = \nabla_a S^{\alpha}{}_{\alpha}{}^{a} + 2\ t_{\alpha}{}^{\alpha} + 2\ L_{(X)} R_{\psi}(\mathbb{1})^{(X)}{}_{(Y)}\ \psi^{(Y)}. \qquad (52)$$

When the matter field equations are satisfied, this becomes the conservation law for dilation current,

$$0 = \nabla_a S^{\alpha}{}_{\alpha}{}^{a} + 2\ t_{\alpha}{}^{\alpha} - 2\ T_{\alpha}{}^{\alpha}. \qquad (53)$$

Now substitute into equation (22), the coordinate variation of L_M, g_{ab}, $g_{\alpha\beta}$, $(\sigma^{-1})^{\alpha}{}_{a}$, $\Gamma^{\alpha}{}_{\beta a}$, and $\psi^{(X)}$ as listed in Table II.9:

$$- \varepsilon^{b}\ \partial_{b}L_{M} + \frac{1}{2} L_{M}\, g^{cd}[-\varepsilon^{b}\ \partial_{b}g_{cd} - (\partial_{c}\varepsilon^{b})g_{bd} - (\partial_{d}\varepsilon^{b})g_{cb}]$$

$$= \frac{1}{2} T^{\alpha\beta}[-\varepsilon^{b}\ \partial_{b}g_{\alpha\beta}] + t_{\alpha}{}^{a}[-\varepsilon^{b}\ \partial_{b}(\sigma^{-1})^{\alpha}{}_{a} - (\partial_{a}\varepsilon^{b})(\sigma^{-1})^{\alpha}{}_{b}]$$

$$+ \frac{1}{2} S^{\beta}{}_{\alpha}{}^{a}[-\varepsilon^{b}\ \partial_{b}\Gamma^{\alpha}{}_{\beta a} - (\partial_{a}\varepsilon^{b})\Gamma^{\alpha}{}_{\beta b}] + L_{(X)}[-\varepsilon^{b}\ \partial_{b}\psi^{(X)}]$$

$$+ \partial_{a}\left(\frac{\partial L_{M}}{\partial\partial_{a}\psi^{(X)}}[-\varepsilon^{b}\ \partial_{b}\psi^{(X)}]\right) + \{^{a}{}_{ca}\}\frac{\partial L_{M}}{\partial\partial_{c}\psi^{(X)}}[-\varepsilon^{b}\ \partial_{b}\psi^{(X)}] \ . \tag{54}$$

(See the note preceding equation (20).) Since equation (54) must be true at all points for all choices of ε^{b}, the coefficients of $-\varepsilon^{b}$ and $-\partial_{a}\varepsilon^{b}$ may be separately equated:

$$\partial_{b}L_{M} + \frac{1}{2} L_{M}\, g^{cd}\ \partial_{b}g_{cd}$$

$$= \frac{1}{2} T^{\alpha\beta}\ \partial_{b}g_{\alpha\beta} + t_{\alpha}{}^{a}\ \partial_{b}(\sigma^{-1})^{\alpha}{}_{a} + \frac{1}{2} S^{\beta}{}_{\alpha}{}^{a}\ \partial_{b}\Gamma^{\alpha}{}_{\beta a} + L_{(X)}\ \partial_{b}\psi^{(X)}$$

$$+ \partial_{a}\left(\frac{\partial L_{M}}{\partial\partial_{a}\psi^{(X)}}\ \partial_{b}\psi^{(X)}\right) + \{^{a}{}_{ca}\}\frac{\partial L_{M}}{\partial\partial_{c}\psi^{(X)}}\ \partial_{b}\psi^{(X)} \ , \tag{55}$$

$$L_{M}\,\delta^{a}{}_{b} = t_{\alpha}{}^{a}\ (\sigma^{-1})^{\alpha}{}_{b} + \frac{1}{2} S^{\beta}{}_{\alpha}{}^{a}\ \Gamma^{\alpha}{}_{\beta b} + \frac{\partial L_{M}}{\partial\partial_{a}\psi^{(X)}}\ \partial_{b}\psi^{(X)} \ . \tag{56}$$

Substituting equation (34) for $S^{\beta}{}_{\alpha}{}^{a}$ into (56) yields

$$t_{b}{}^{a} = L_{M}\,\delta^{a}{}_{b} - \frac{\partial L_{M}}{\partial\partial_{a}\psi^{(X)}}\left(\partial_{b}\psi^{(X)} + \Gamma^{\alpha}{}_{\beta b}\ R_{\psi}(E^{\beta}{}_{\alpha})^{(X)}{}_{(Y)}\ \psi^{(Y)}\right)$$

$$= L_{M}\,\delta^{a}{}_{b} - \frac{\partial L_{M}}{\partial\partial_{a}\psi^{(X)}}\,\nabla_{b}\psi^{(X)} \ , \tag{57}$$

which is the definition of the canonical energy-momentum tensor used in particle physics.

Equation (56) may now be substituted into equation (55) to produce

$$\partial_b L_M + \frac{1}{2} L_M g^{cd} \partial_b g_{cd}$$

$$= \frac{1}{2} T^{\alpha\beta} \partial_b g_{\alpha\beta} + t_\alpha{}^a \partial_b (\sigma^{-1})^\alpha{}_a + \frac{1}{2} S^\beta{}_\alpha{}^a \partial_b \Gamma^\alpha{}_{\beta a} + L_{(X)} \partial_b \psi^{(X)}$$

$$+ \partial_a [L_M \delta^a{}_b - t_\alpha{}^a (\sigma^{-1})^\alpha{}_b - \frac{1}{2} S^\beta{}_\alpha{}^a \Gamma^\alpha{}_{\beta b}]$$

$$+ \{^a{}_{ca}\} [L_M \delta^c{}_b - t_\alpha{}^c (\sigma^{-1})^\alpha{}_b - \frac{1}{2} S^\beta{}_\alpha{}^c \Gamma^\alpha{}_{\beta b}] \ .$$

Since $\{^a{}_{ba}\} = \frac{1}{2} g^{cd} \partial_b g_{cd}$, the L_M terms cancel leaving

$$0 = \frac{1}{2} T^{\alpha\beta} \partial_b g_{\alpha\beta} + L_{(X)} \partial_b \psi^{(X)}$$

$$+ t_\alpha{}^a [\partial_b (\sigma^{-1})^\alpha{}_a - \partial_a (\sigma^{-1})^\alpha{}_b] - (\sigma^{-1})^\alpha{}_b [\partial_a t_\alpha{}^a + \{^a{}_{ca}\} t_\alpha{}^c]$$

$$+ \frac{1}{2} S^\beta{}_\alpha{}^a [\partial_b \Gamma^\alpha{}_{\beta a} - \partial_a \Gamma^\alpha{}_{\beta b}] - \frac{1}{2} \Gamma^\alpha{}_{\beta b} [\partial_a S^\beta{}_\alpha{}^a + \{^a{}_{ca}\} S^\beta{}_\alpha{}^c] \ .$$

Now convert all partial derivatives into covariant derivatives.

$$0 = \frac{1}{2} T^{\alpha\beta} [\nabla_b g_{\alpha\beta} + \Gamma^\gamma{}_{\alpha b} g_{\gamma\beta} + \Gamma^\gamma{}_{\beta b} g_{\alpha\gamma}]$$

$$+ L_{(X)} [\nabla_b \psi^{(X)} - \Gamma^\alpha{}_{\beta b} R_\psi (E^\beta{}_\alpha)^{(X)}{}_{(Y)} \psi^{(Y)}]$$

$$+ t_\alpha{}^a [\nabla_b (\sigma^{-1})^\alpha{}_a - \nabla_a (\sigma^{-1})^\alpha{}_b - \Gamma^\alpha{}_{\beta b} (\sigma^{-1})^\beta{}_a + \Gamma^\alpha{}_{\beta a} (\sigma^{-1})^\beta{}_b]$$

$$- (\sigma^{-1})^\alpha{}_b [\nabla_a t_\alpha{}^a + \Gamma^\beta{}_{\alpha a} t_\beta{}^a]$$

$$+ \frac{1}{2} S^\beta{}_\alpha{}^a [\hat{R}^\alpha{}_{\beta b a} - \Gamma^\alpha{}_{\gamma b} \Gamma^\gamma{}_{\beta a} + \Gamma^\alpha{}_{\gamma a} \Gamma^\gamma{}_{\beta b}]$$

$$- \frac{1}{2} \Gamma^\alpha{}_{\beta b} [\nabla_a S^\beta{}_\alpha{}^a - \Gamma^\beta{}_{\gamma a} S^\gamma{}_\alpha{}^a + \Gamma^\gamma{}_{\alpha a} S^\beta{}_\gamma{}^a] \ . \qquad (58)$$

Equation (III.3.7) implies

$$\nabla_b(\sigma^{-1})^{\alpha}{}_a - \nabla_a(\sigma^{-1})^{\alpha}{}_b = (\sigma^{-1})^{\alpha}{}_c(\lambda^c{}_{ab} - \lambda^c{}_{ba}) = Q^{\alpha}{}_{ba} . \tag{59}$$

Hence (58) becomes

$$0 = -(\sigma^{-1})^{\alpha}{}_b \nabla_a t_{\alpha}{}^a + \frac{1}{2} S^{\beta}{}_{\alpha}{}^a \hat{R}^{\alpha}{}_{\beta ba} + t_{\alpha}{}^a Q^{\alpha}{}_{ba}$$
$$+ \frac{1}{2} T^{\alpha\beta} \nabla_b g_{\alpha\beta} + L_{(X)} \nabla_b \psi^{(X)}$$
$$+ \Gamma^{\alpha}{}_{\beta b} [T_{\alpha}{}^{\beta} - \frac{1}{2} \nabla_a S^{\beta}{}_{\alpha}{}^a - t_{\alpha}{}^{\beta} - L_{(X)} R_{\psi}(E^{\beta}{}_{\alpha})^{(X)}{}_{(Y)} \psi^{(Y)}] .$$

The coefficient of $\Gamma^{\alpha}{}_{\beta b}$ vanishes by virtue of equation (45) leaving

$$(\sigma^{-1})^{\alpha}{}_b \nabla_a t_{\alpha}{}^a = \frac{1}{2} S^{\beta}{}_{\alpha}{}^a \hat{R}^{\alpha}{}_{\beta ba} + t_{\alpha}{}^a Q^{\alpha}{}_{ba} + \frac{1}{2} T^{\alpha\beta} \nabla_b g_{\alpha\beta} + L_{(X)} \nabla_b \psi^{(X)} . \tag{60}$$

When the matter field equations, $L_{(X)} = 0$, are satisfied this becomes the conservation law for energy-momentum,

$$(\sigma^{-1})^{\alpha}{}_b \nabla_a t_{\alpha}{}^a = \frac{1}{2} S^{\beta}{}_{\alpha}{}^a \hat{R}^{\alpha}{}_{\beta ba} + t_{\alpha}{}^a Q^{\alpha}{}_{ba} + \frac{1}{2} T^{\alpha\beta} \nabla_b g_{\alpha\beta}, \tag{61}$$

or

$$\nabla_a t_{\gamma}{}^a = \frac{1}{2} S^{\beta}{}_{\alpha}{}^{\delta} \hat{R}^{\alpha}{}_{\beta\gamma\delta} + t_{\alpha}{}^{\delta} Q^{\alpha}{}_{\gamma\delta} + \frac{1}{2} T^{\alpha\beta} \sigma^b{}_{\gamma} \nabla_b g_{\alpha\beta} , \tag{62}$$

or

$$\nabla_a t_b{}^a = \frac{1}{2} S^d{}_c{}^a \hat{R}^c{}_{dba} + (t_c{}^d - T_c{}^d)\lambda^c{}_{db} , \tag{63}$$

or

$$\sigma^a{}_{\delta} \nabla_a t_{\gamma}{}^{\delta} = \frac{1}{2} S^{\beta}{}_{\alpha}{}^{\delta} \hat{R}^{\alpha}{}_{\beta\gamma\delta} + t_{\alpha}{}^{\delta} Q^{\alpha}{}_{\gamma\delta} + t_{\gamma}{}^{\delta} \lambda^{\alpha}{}_{\delta\gamma} + \frac{1}{2} T^{\alpha\beta} \sigma^b{}_{\gamma} \nabla_b g_{\alpha\beta} . \tag{64}$$

I have written the conservation law in four forms. Equations (61) and (62) have mixed covariant derivatives; equation (63) has all external Christoffel covariant derivatives; while equation (64) has all internal full covariant derivatives.

c. Conservation Laws with an $O_\circ(3,1,R)$-Internal Tangent Frame Bundle or an $SL(2,C)$-Spinor Frame Bundle

The $O_\circ(3,1,R)$-internal tangent frame bundle, $O_\circ(M,g)$, consists of all oriented, time oriented, orthonormal frames at all points of spacetime. Its connection must be an $O_\circ(3,1,R)$-connection; i.e. the connection coefficients, $\Gamma^\alpha{}_{\beta a}$, are written in orthonormal internal frames and take their values in $\mathcal{L}O_\circ(3,1,R)$. Hence, $\Gamma_{\alpha\beta a}$ is antisymmetric in α and β and the connection is metric-compatible, although not necessarily torsion-free; i.e. a Cartan connection.

The antisymmetry of $\Gamma_{\alpha\beta a}$ implies $S^{\beta\alpha a}$ is also antisymmetric in α and β. A transformation between orthonormal frames leaves the metric invariant. Hence, under infinitesimal variations, $\lambda_{\alpha\beta}$ is antisymmetric.

The matter fields, $\psi^{(X)}$, may have spinor indices, but in that case there must be a spinor frame bundle to which $O_\circ(M,g)$ is associated. In this subsection I take that spinor frame bundle to be the $SL(2,C)$-spinor frame bundle, $SL_{Spin}(M)$, which consists of all orthonormal spinor frames at all points of spacetime. In this case, $\Gamma^\alpha{}_{\beta a}$, is actually an $SL(2,C)$-connection.

The results are also applicable to $O(3,1,R)$, $SO(3,1,R)$, $O_T(3,1,R)$, $O_S(3,1,R)$, $SL'(2,C)$, $SL_+(2,C)$, $SL_T(2,C)$, and $SL_S(2,C)$ which all have the same Lie algebra as $SL(2,C)$ and $O_\circ(3,1,R)$. Unlike the $GL_\circ(4,R)$ case, there is no hypermomentum, $S_{(\beta\alpha)}{}^a$, and hence no dilation current, $S^\alpha{}_\alpha{}^a$. Correspondingly there are no conservation laws for hypermomentum or dilation current.

Substitute into equation (22), the $O_0(3,1,R)$-internal tangent frame (or SL(2,C)-spinor frame) variations of L_M, g_{ab}, $g_{\alpha\beta}$, $(\sigma^{-1})^{\alpha}{}_{a}$, $\Gamma^{\alpha}{}_{\beta a}$, and $\psi^{(X)}$, as listed in Tables II.10 and II.11:

$$0 = t_{\alpha}{}^{a}\,\lambda^{\alpha}{}_{\beta}(\sigma^{-1})^{\beta}{}_{a} - \frac{1}{2}\,S^{\beta}{}_{\alpha}{}^{a}\;\nabla_a\lambda^{\alpha}{}_{\beta}$$
$$+ L_{(X)}\,\frac{1}{2}\,\lambda^{\alpha}{}_{\beta}\,R_{\psi}(\sigma^{\beta}{}_{\alpha})^{(X)}{}_{(Y)}\;\psi^{(Y)} + \nabla_a\left(\frac{\partial L_M}{\partial\partial_a\psi^{(X)}}\,\frac{1}{2}\,\lambda^{\alpha}{}_{\beta}R_{\psi}(\sigma^{\beta}{}_{\alpha})^{(X)}{}_{(Y)}\psi^{(Y)}\right), \tag{65}$$

or

$$0 = \lambda^{\alpha}{}_{\beta}\left[t_{\alpha}{}^{\beta} + \frac{1}{2}\,L_{(X)}\;R_{\psi}(\sigma^{\beta}{}_{\alpha})^{(X)}{}_{(Y)}\;\psi^{(Y)} + \frac{1}{2}\,\nabla_a\left(\frac{\partial L_M}{\partial\partial_a\psi^{(X)}}\,R_{\psi}(\sigma^{\beta}{}_{\alpha})^{(X)}{}_{(Y)}\;\psi^{(Y)}\right)\right]$$
$$+ \nabla_a\lambda^{\alpha}{}_{\beta}\left[-\frac{1}{2}\,S^{\beta}{}_{\alpha}{}^{a} + \frac{1}{2}\,\frac{\partial L_M}{\partial\partial_a\psi^{(X)}}\,R_{\psi}(\sigma^{\beta}{}_{a})^{(X)}{}_{(Y)}\;\psi^{(Y)}\right]. \tag{66}$$

Since this equation must be true at all points for all antisymmetric choices of $\lambda^{\alpha}{}_{\beta}$, the coefficients of $\lambda^{\alpha}{}_{\beta}$ and $\nabla_a\lambda^{\alpha}{}_{\beta}$ may be separately antisymmetrized and equated to zero:

$$0 = 2\,t_{[\alpha\beta]} + L_{(X)}\;R_{\psi}(\sigma_{\beta\alpha})^{(X)}{}_{(Y)}\;\psi^{(Y)}$$
$$+ \nabla_a\left(\frac{\partial L_M}{\partial\partial_a\psi^{(X)}}\;R_{\psi}(\sigma_{\beta\alpha})^{(X)}{}_{(Y)}\;\psi^{(Y)}\right), \tag{67}$$

$$S_{\beta\alpha}{}^{a} = \frac{\partial L_M}{\partial\partial_a\psi^{(X)}}\,R_{\psi}(\sigma_{\beta\alpha})^{(X)}{}_{(Y)}\;\psi^{(Y)}\quad. \tag{68}$$

In deriving these equations recall that $S_{\beta\alpha}{}^{a}$ and $\sigma_{\beta\alpha}$ are already antisymmetric.

As in the $GL_0(4,R)$ case, equation (68) is an alternate definition of the canonical spin (angular momentum) tensor. (Compare (68) with (34) and (41).) Equating (68) with the original definition, equation (15) or (35), yields,

$$\frac{\partial L_M}{\partial \Gamma^{\alpha}{}_{\beta a}} = \frac{1}{2} \frac{\partial L_M}{\partial \partial_a \psi^{(X)}} R_\psi(\sigma^\beta{}_\alpha)^{(X)}{}_{(Y)} \psi^{(Y)} . \tag{69}$$

As in the GL∘(4,R) case, this implies that L_M is minimally coupled; i.e. L_M depends on $\partial_a \psi^{(X)}$ and on $\Gamma^\alpha{}_{\beta a}$ only through the covariant derivative

$$\nabla_a \psi^{(X)} = \partial_a \psi^{(X)} + \frac{1}{2} \Gamma^\alpha{}_{\beta a} R_\psi(\sigma^\beta{}_\alpha)^{(X)}{}_{(Y)} \psi^{(Y)}. \tag{70}$$

Equation (68) may be substituted into equation (67) yielding,

$$0 = \nabla_a S_{\beta\alpha}{}^a + 2\, t_{[\alpha\beta]} + L_{(X)} R_\psi(\sigma_{\beta\alpha})^{(X)}{}_{(Y)} \psi^{(Y)}. \tag{71}$$

When the matter field equations, $L_{(X)} = 0$, are satisfied, this becomes the conservation law for angular momentum,

$$0 = \nabla_a S_{\beta\alpha}{}^a + 2\, t_{[\alpha\beta]}. \tag{72}$$

Now substitute into equation (22), the coordinate variations of L_M, g_{ab}, $g_{\alpha\beta}$, $(\sigma^{-1})^\alpha{}_a$, $\Gamma^\alpha{}_{\beta a}$, and $\psi^{(X)}$ as listed in Table II.9:

$$\begin{aligned} &- \varepsilon^b \partial_b L_M + \frac{1}{2} L_M g^{cd}[- \varepsilon^b \partial_b g_{cd} - (\partial_c \varepsilon^b) g_{bd} - (\partial_d \varepsilon^b) g_{cb}] \\ &= t_\alpha{}^a [-\varepsilon^b \partial_b (\sigma^{-1})^\alpha{}_a - (\partial_a \varepsilon^b)(\sigma^{-1})^\alpha{}_b] \\ &\quad + \frac{1}{2} S^\beta{}_\alpha{}^a [-\varepsilon^b \partial_b \Gamma^\alpha{}_{\beta a} - (\partial_a \varepsilon^b) \Gamma^\alpha{}_{\beta b}] + L_{(X)} [-\varepsilon^b \partial_b \psi^{(X)}] \\ &\quad + \partial_a \left(\frac{\partial L_M}{\partial \partial_a \psi^{(X)}} [-\varepsilon^b \partial_b \psi^{(X)}] \right) + \{^a{}_{ca}\} \frac{\partial L_M}{\partial \partial_c \psi^{(X)}} [-\varepsilon^b \partial_b \psi^{(X)}]. \end{aligned} \tag{73}$$

Equate the coefficients of $-\varepsilon^b$ and of $-\partial_a \varepsilon^b$,

$$\partial_b L_M + \frac{1}{2} L_M g^{cd} \partial_b g_{cd}$$

$$= t_\alpha{}^a \partial_b (\sigma^{-1})^\alpha{}_a + \frac{1}{2} S^\beta{}_\alpha{}^a \partial_b \Gamma^\alpha{}_{\beta a} + L_{(X)} \partial_b \psi^{(X)}$$

$$+ \partial_a \left(\frac{\partial L_M}{\partial \partial_a \psi^{(X)}} \partial_b \psi^{(X)} \right) + \{^a{}_{ca}\} \frac{\partial L_M}{\partial \partial_c \psi^{(X)}} \partial_b \psi^{(X)} , \qquad (74)$$

$$L_M \delta^a{}_b = t_\alpha{}^a (\sigma^{-1})^\alpha{}_b + \frac{1}{2} S^\beta{}_\alpha{}^a \Gamma^\alpha{}_{\beta b} + \frac{\partial L_M}{\partial \partial_a \psi^{(X)}} \partial_b \psi^{(X)} . \qquad (75)$$

Using equation (68) for $S^\beta{}_\alpha{}^a$, equation (75) becomes

$$t_b{}^a = L_M \delta^a{}_b - \frac{\partial L_M}{\partial \partial_a \psi^{(X)}} \nabla_b \psi^{(X)} , \qquad (76)$$

which is again the definition of the canonical energy-momentum tensor used in particle physics. Substituting (75) into (74) and cancelling the L_M terms leaves,

$$0 = t_\alpha{}^a [\partial_b (\sigma^{-1})^\alpha{}_a - \partial_a (\sigma^{-1})^\alpha{}_b] - (\sigma^{-1})^\alpha{}_b [\partial_a t_\alpha{}^a + \{^a{}_{ca}\} t_\alpha{}^c]$$

$$+ \frac{1}{2} S^\beta{}_\alpha{}^a [\partial_b \Gamma^\alpha{}_{\beta a} - \partial_a \Gamma^\alpha{}_{\beta b}] - \frac{1}{2} \Gamma^\alpha{}_{\beta b} [\partial_a S^\beta{}_\alpha{}^a + \{^a{}_{ca}\} S^\beta{}_\alpha{}^c]$$

$$+ L_{(X)} \partial_b \psi^{(X)} . \qquad (77)$$

Convert to covariant derivatives and use equation (59).

$$0 = - (\sigma^{-1})^\alpha{}_b \nabla_a t_\alpha{}^a + \frac{1}{2} S^\beta{}_\alpha{}^a \hat{R}^\alpha{}_{\beta ba} + t_\alpha{}^a Q^\alpha{}_{ba} + L_{(X)} \nabla_b \psi^{(X)}$$

$$- \Gamma^\alpha{}_{\beta b} [t_\alpha{}^\beta + \frac{1}{2} \nabla_a S^\beta{}_\alpha{}^a + \frac{1}{2} L_{(X)} R_\psi (\sigma^\beta{}_\alpha)^{(X)}{}_{(Y)} \psi^{(Y)}] . \qquad (78)$$

Since $\Gamma^{\alpha}{}_{\beta b}$, $S^{\beta}{}_{\alpha}{}^{a}$ and $\sigma^{\beta}{}_{\alpha}$ are antisymmetric in α and β, the coefficient of $\Gamma^{\alpha}{}_{\beta b}$ vanishes by virtue of equation (71), leaving

$$(\sigma^{-1})^{\alpha}{}_{b}\, \nabla_{a} t_{\alpha}{}^{a} = \frac{1}{2} S^{\beta}{}_{\alpha}{}^{a}\, \hat{R}^{\alpha}{}_{\beta ba} + t_{\alpha}{}^{a}\, Q^{\alpha}{}_{ba} + L_{(X)}\, \nabla_{b}\psi^{(X)} \ . \tag{79}$$

When the matter field equations, $L_{(X)} = 0$, are satisfied, this becomes the conservation law for energy-momentum,

$$(\sigma^{-1})^{\alpha}{}_{b}\, \nabla_{a} t_{\alpha}{}^{a} = \frac{1}{2} S^{\beta}{}_{\alpha}{}^{a}\, \hat{R}^{\alpha}{}_{\beta ba} + t_{\alpha}{}^{a}\, Q^{\alpha}{}_{ba}, \tag{80}$$

or

$$\nabla_{a} t_{\gamma}{}^{a} = \frac{1}{2} S^{\beta}{}_{\alpha}{}^{\delta}\, \hat{R}^{\alpha}{}_{\beta\gamma\delta} + t_{\alpha}{}^{\delta}\, Q^{\alpha}{}_{\gamma\delta} \ , \tag{81}$$

or

$$\nabla_{a} t_{b}{}^{a} = \frac{1}{2} S^{d}{}_{c}{}^{a}\, \hat{R}^{c}{}_{dba} + t_{c}{}^{d}\, \lambda^{c}{}_{db} \ , \tag{82}$$

or

$$\sigma^{a}{}_{\delta}\, \nabla_{a} t_{\gamma}{}^{\delta} = \frac{1}{2} S^{\beta}{}_{\alpha}{}^{\delta}\, \hat{R}^{\alpha}{}_{\beta\gamma\delta} + t_{\alpha}{}^{\delta}\, Q^{\alpha}{}_{\gamma\delta} + t_{\gamma}{}^{\delta}\, \lambda^{\alpha}{}_{\delta\alpha} \ . \tag{83}$$

Equations (80) and (81) use mixed covariant derivatives; equation (82) uses external Christoffel covariant derivatives; while equation (83) uses internal full covariant derivatives.